Section 7.2 Asymptotic Growth of Functions

Notation

• Z^+ denotes the set of positive integers

•
$$Z^+ = \{1, 2, 3, ...\}$$

- R^{\geq} denotes the set of real numbers greater than or equal to 0
 - $\mathbf{R}^{\geq} = \{n \mid n \in \mathbf{R} \text{ and } n \geq 0\}$

Big- \mathcal{O} Notation

- Let f and g be functions from the set Z^+ to the set R^{\geq}
 - $f: \mathbb{Z}^+ \to \mathbb{R}^{\geq}$
 - g: $Z^+ \rightarrow R^{\geq}$
- f(n) is O(g(n)) if there are positive real constants c and n_0 such that $f(n) \le c \cdot g(n)$

whenever $n \ge n_0$

Such c and n_0 are called witnesses to the claim that f(n) is O(g(n))

• Example: Show that $7n^2$ is $\mathcal{O}(n^3)$

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- Example: Show that $7n^2$ is $\mathcal{O}(n^3)$
 - $7n^2 \le n \cdot n^2$ when $n \ge 7$ $7n^2 \le 1 \cdot n^3$ when $n \ge 7$

• Thus $7n^2$ is $\mathcal{O}(n^3)$ with witnesses c = 1 and $n_0 = 7$

• Another example: Show that $f(n) = n^2 + 2n + 1$ is $O(n^2)$

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 $2n \le n^2$ when $n \ge 2$

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$$n^{2} \leq n^{2}$$

$$2n \leq n^{2} \quad \text{when } n \geq 2$$

$$1 \leq n^{2} \quad \text{when } n \geq 1$$

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- Note that:

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$$2n \leq n^{2} \quad \text{when } n \geq 2$$

$$1 \leq n^{2} \quad \text{when } n \geq 1$$

$$n^{2} + 2n + 1 \leq 3n^{2} \quad \text{when } n \geq \max(2, 1) = 2$$

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$$n^{2} + 2n + 1 \leq 3n^{2} \quad \text{when } n \geq \max(2, 1) = 2$$

$$n^2 + 2n + 1$$
 is $\mathcal{O}(n^2)$ with witnesses $c = 3$ and $n_0 = 2$

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- Note that:

 $3n^2 \le 3n^2$ $2n \le 2n^2$ when $n \ge 1$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $O(n^2)$
- Note that:

 $3n^{2} \leq 3n^{2}$ $2n \leq 2n^{2} \text{ when } n \geq 1$ $4 \leq 4n^{2} \text{ when } n \geq 1$

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- Note that:

 $3n^{2} \leq 3n^{2}$ $2n \leq 2n^{2} \text{ when } n \geq 1$ $4 \leq 4n^{2} \text{ when } n \geq 1$ $3n^{2} + 2n + 4 \leq 9n^{2} \text{ when } n \geq 1$

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- Note that:

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 $3n^2 + 2n + 1$ is $\mathcal{O}(n^2)$ with witnesses c = 9 and $n_0 = 1$

• Still another example: Show that $f(n) = 1 + 2 + \cdots n$ is $O(n^2)$

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$$f(n) = 1 + 2 + \dots n \le n + n + \dots + n$$

n times

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$$f(n) = 1 + 2 + \dots n \le n + n + \dots + n = n^2$$

$$n \text{ times}$$

• Still another example: Show that $f(n) = 1 + 2 + \cdots n$ is $\mathcal{O}(n^2)$

$$f(n) = 1 + 2 + \dots n \le n + n + \dots + n = n^2$$

n times

 $f(n) = 1 + 2 + \cdots n$ is $O(n^2)$ with witnesses c = 1 and $n_0 = 1$

$$f(n) = 1 \cdot 2 \cdot \cdots \cdot n$$

$$f(n) = 1 \cdot 2 \cdot \dots \cdot n \le n \cdot n \cdot \dots \cdot n$$
n times

$$f(n) = 1 \cdot 2 \cdot \dots \cdot n \le n \cdot n \cdot \dots \cdot n = n^n$$
n times

Showing that
$$f(n)$$
 is $\mathcal{O}(g(n))$

• And another example: Show that $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$

$$f(n) = 1 \cdot 2 \cdot \dots \cdot n \leq n \cdot n \cdot \dots \cdot n = n^{n}$$

$$n \text{ times}$$

 $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$ with witnesses c = 1 and $n_0 = 1$

• To show that f(n) is not $\mathcal{O}(g(n))$, you must show that for any c and n_0 , it is not the case that $f(n) \leq c \cdot g(n)$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction

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 - 1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0
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 - 3. $n^2 \le c \cdot n$ for all n when $n \ge n_0$, n > 0, and n > c

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 - 5. $n \le c$ when n > c is a contradiction

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 - 3. $n^3 \le c \cdot 7n^2$ when $n \ge n_0$, n > 0, and $n > c \cdot 7$

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 when $n \ge n_0$, $n > 0$, and $n > c \cdot 7$

- 4. $n \le c \cdot 7$ when $n \ge n_0$, n > 0, and $n > c \cdot 7$
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- 4. $n \le c \cdot 7$ when $n \ge n_0$, n > 0, and $n > c \cdot 7$
- 5. $n \le c \cdot 7$ when $n > c \cdot 7$ is a contradiction
- 6. n^3 is not $\mathcal{O}(7n^2)$

- Let f and g be functions from the set Z^+ to the set R^{\geq}
 - $f: \mathbb{Z}^+ \to \mathbb{R}^{\geq}$
 - g: $Z^+ \to R^{\geq}$
- f(n) is $\Omega(g(n))$ if there are <u>positive</u> real constants c and n_0 such that $f(n) \ge c \cdot g(n)$

whenever $n \ge n_0$

Such c and n_0 are called witnesses to the claim that f(n) is $\Omega(g(n))$

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 - iff there are positive real 1/c and n_0 such that $g(n) \ge \frac{1}{c} \cdot f(n)$ when $n \ge n_0$

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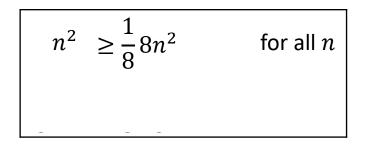
iff there are positive real c and n_0 such that $g(n) \ge c \cdot f(n)$ when $n \ge n_0$ iff g is $\Omega(f)$

- Example: $f(n) = 8n^3 + 5n^2 + 7$ is $\Omega(g)$ where $g(n) = n^3$
- Proof:

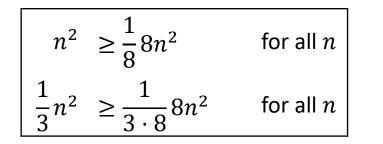
 $8n^3 + 5n^2 + 7 \ge n^3$ whenever n > 0So, $8n^3 + 5n^2 + 7 \ge n^3$ is $\Omega(g)$ with witnesses c = 1 and $n_0 = 0$

• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g(n))$ where $g(n) = 8n^2$

- Another example: $f(n) = n^2 4n 2$ is $\Omega(g)$ where $g(n) = 8n^2$
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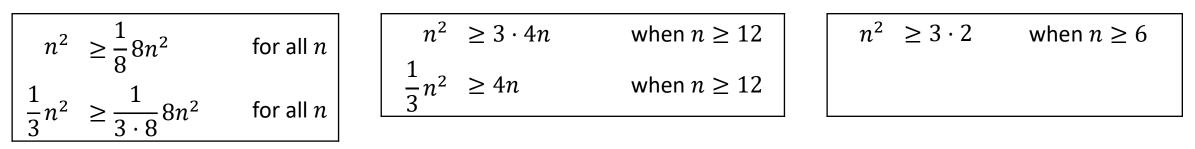
• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$

$n^2 \ge \frac{1}{8}8n^2$	for all <i>n</i>	$n^2 \geq 3 \cdot 4n$	when $n \ge 12$
$\left \begin{array}{c} \frac{1}{3}n^2 \ge \frac{1}{3\cdot 8}8n^2 \end{array}\right.$	for all <i>n</i>		

• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$

$n^2 \geq \frac{1}{2} 8n^2$	for all n	$n^2 \geq 3 \cdot 4n$	when $n \ge 12$
0	for all <i>n</i>	$\frac{1}{3}n^2 \ge 4n$	when $n \ge 12$
$3^{n} = 3 \cdot 8^{0n}$			

• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$

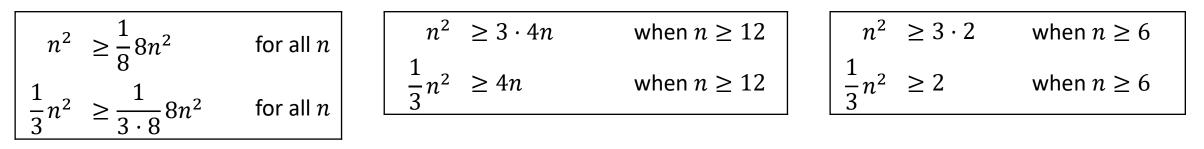


• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$

$n^2 \geq \frac{1}{2}8n^2$	for all n	$n^2 \geq 3 \cdot 4n$	when $n \ge 12$	$n^2 \geq 3 \cdot 2$	when $n \ge 6$
$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -8 \\ -8 \end{bmatrix}$		$\frac{1}{3}n^2 \ge 4n$	when $n \ge 12$	$\left \frac{1}{3}n^2 \right \geq 2$	when $n \ge 6$
$\left \frac{1}{3}n^2 \right \geq \frac{1}{3 \cdot 8}8n^2$	for all n	5		5	

• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$

• Proof:

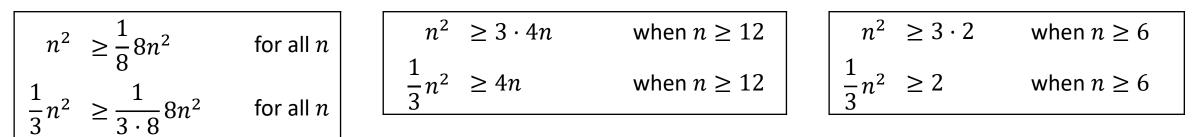


when $n \ge 6$

$$\frac{1}{3}n^2 + \frac{1}{3}n^2 + \frac{1}{3}n^2 \ge \frac{1}{3 \cdot 8}8n^2 + 4n + 2 \qquad \text{when } n \ge 12 \text{ and}$$

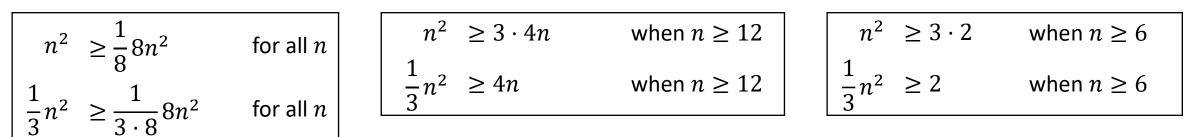
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• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$



$$\frac{1}{3}n^{2} + \frac{1}{3}n^{2} + \frac{1}{3}n^{2} \ge \frac{1}{3 \cdot 8}8n^{2} + 4n + 2 \qquad \text{when } n \ge 12 \text{ and when } n \ge 6$$
$$n^{2} \ge \frac{1}{3 \cdot 8}8n^{2} + 4n + 2 \qquad \text{when } n \ge 12 \text{ and when } n \ge 6$$

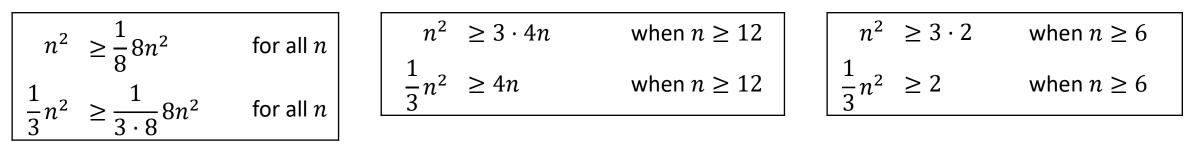
• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$



$$\frac{1}{3}n^{2} + \frac{1}{3}n^{2} + \frac{1}{3}n^{2} \ge \frac{1}{3 \cdot 8}8n^{2} + 4n + 2 \qquad \text{when } n \ge 12 \text{ and when } n \ge 6$$
$$n^{2} \ge \frac{1}{3 \cdot 8}8n^{2} + 4n + 2 \qquad \text{when } n \ge 12 \text{ and when } n \ge 6$$
$$n^{2} - 4n - 2 \ge \frac{1}{3 \cdot 8}8n^{2} \qquad \text{when } n \ge 12$$

• Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$

• Proof:



$$\frac{1}{3}n^{2} + \frac{1}{3}n^{2} + \frac{1}{3}n^{2} \ge \frac{1}{3 \cdot 8}8n^{2} + 4n + 2 \qquad \text{when } n \ge 12 \text{ and when } n \ge 6$$
$$n^{2} \ge \frac{1}{3 \cdot 8}8n^{2} + 4n + 2 \qquad \text{when } n \ge 12 \text{ and when } n \ge 6$$
$$n^{2} - 4n - 2 \ge \frac{1}{3 \cdot 8}8n^{2} \qquad \text{when } n \ge 12$$

 $n^2 - 4n - 2$ is $\Omega(8n^2)$ with witnesses $c = \frac{1}{24}$ and $n_0 = 12$

- Let *f* and *g* be functions from either the set of integers or the set of real numbers to the set of real numbers
- f(n) is $\Theta(g(n))$ if f(n) is O(g(n)) and f(n) is $\Omega(g(n))$

Theorem: If f(n) is $\Theta(g(n))$ then g(n) is $\Theta(f(n))$

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Theorem: If f(n) is $\Theta(g(n))$ then g(n) is $\Theta(f(n))$

- 1. Assume f(n) is $\Theta(g(n))$
- 2. f(n) is O(g(n)) and f(n) is $\Omega(g(n))$
- 3. g(n) is $\Omega(f(n))$ and g(n) is O(g(n)) (proved last lecture)

$\mathsf{Big}\text{-}\Theta\ \mathsf{Notation}$

Theorem: If f(n) is $\Theta(g(n))$ then g(n) is $\Theta(f(n))$

- 1. Assume f(n) is $\Theta(g(n))$
- 2. f(n) is O(g(n)) and f(n) is $\Omega(g(n))$
- 3. g(n) is $\Omega(f(n))$ and g(n) is O(g(n)) (proved last lecture) 4. g(n) is $\Theta(f(n))$

Big-O Notation

- Example: Show that $f(n) = 1 + 2 + \dots + n$ is $\Theta(n^2)$
- Proof: A previous example showed that $f(n) = 1 + 2 + \dots + n$ is $O(n^2)$. To show that f(n) is $O(n^2)$, we just need to show that f(n) is $\Omega(n^2)$

 $f(n) = 1 + 2 + \dots + (n - 1) + n$

$$f(n) = 1 + 2 + \dots + (n - 1) + n$$

 $2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$ $+ 1 + 2 + \dots + (n - 1) + n$

$$f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n + 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n-1) + n + n + (n-1) + \dots + 2 + 1$$

$$f(n) = 1 + 2 + \dots + (n - 1) + n$$

 $2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$ $+ 1 + 2 + \dots + (n - 1) + n$

$$2 \cdot f(n) = 1 + 2 + \dots + (n-1) + n + n + (n-1) + \dots + 2 + 1$$

 $2 \cdot f(n) = n(n+1)$

$$f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$+ 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$+ n + (n - 1) + \dots + 2 + 1$$

$$2 \cdot f(n) = n(n + 1)$$

$$2 \cdot f(n) = n^{2} + n$$

$$f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$+ 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$+ n + (n - 1) + \dots + 2 + 1$$

$$2 \cdot f(n) = n(n + 1)$$

$$2 \cdot f(n) = n^{2} + n$$

$$f(n) = \frac{1}{2}n^{2} + \frac{1}{2}n$$

$$f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$+ 1 + 2 + \dots + (n - 1) + n$$

$$2 \cdot f(n) = 1 + 2 + \dots + (n - 1) + n$$

$$+ n + (n - 1) + \dots + 2 + 1$$

$$2 \cdot f(n) = n(n + 1)$$

$$2 \cdot f(n) = n^{2} + n$$

$$f(n) = \frac{1}{2}n^{2} + \frac{1}{2}n$$

$$f(n) \ge \frac{1}{2}n^{2} \text{ when } n \ge 1$$

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$$f(n) \ge \frac{1}{2}n^{2} \text{ when } n \ge 1$$

$$f(n)$$
 is $\Omega(n^2)$ with witnesses $c = \frac{1}{2}$ and $n_0 = 1$.
Since $f(n)$ is also $O(n^2)$, $f(n)$ is $\Theta(n^2)$

- Another example: Show that $3n^2 + 8n \cdot \log(n)$ is $\Theta(n^2)$
- Note that when n > 0, $\log(n) < n$

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- Note that when n > 0, $\log(n) < n$

 $\log(n) < n$ when $n \ge 1$

Big-O Notation

- Another example: Show that $3n^2 + 8n \cdot \log(n)$ is $\Theta(n^2)$
- Note that when n > 0, $\log(n) < n$

 $\log(n) < n \qquad \text{when } n \ge 1$ $8n \cdot \log(n) \le 8n^2 \qquad \text{when } n \ge 1$

- Another example: Show that $3n^2 + 8n \cdot \log(n)$ is $\Theta(n^2)$
- Note that when n > 0, $\log(n) < n$

$\log(n)$	< n	when $n \ge 1$
$8n \cdot \log(n)$	$\leq 8n^2$	when $n \geq 1$
$3n^2 + 8n \cdot \log(n)$	$\leq 11n^2$	when $n \ge 1$

- Another example: Show that $3n^2 + 8n \cdot \log(n)$ is $\Theta(n^2)$
- Note that when n > 0, $\log(n) < n$

$\log(n) \leq n$	when $n \ge 1$
$8n \cdot \log(n) \leq 8n^2$	when $n \geq 1$
$3n^2 + 8n \cdot \log(n) \leq 11n^2$	when $n \ge 1$

So, $3n^2 + 8n \cdot \log(n)$ is $\mathcal{O}(n^2)$ with witnesses c = 11 and $n_0 = 1$

• Another example continued

Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

• Another example continued

Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

 $\log(n) \ge 0$ when $n \ge 1$

• Another example continued

Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

$$log(n) \ge 0$$
when $n \ge 1$ $8n \cdot log(n) \ge 0$ when $n \ge 1$

• Another example continued

Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

$\log(n)$	≥ 0	when $n \geq 1$
$8n \cdot \log(n)$	≥ 0	when $n \ge 1$
$3n^2 + 8n \cdot \log(n)$	$\geq n^2$	when $n \geq 1$

• Another example continued

Now we need to show that $3n^2 + 8n \cdot \log(x)$ is $\Omega(n^2)$

$\log(n)$	≥ 0	when $n \ge 1$
$8n \cdot \log(n)$	≥ 0	when $n \geq 1$
$3n^2 + 8n \cdot \log(n)$	$\geq n^2$	when $n \ge 1$

Hence $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$ with witnesses c = 1 and $n_0 = 1$ Since $3n^2 + 8n \cdot \log(n)$ is also $\mathcal{O}(n^2)$, $3n^2 + 8n \cdot \log(n)$ is $\Theta(n^2)$

Asymptotic Growth of Polynomials

• Theorem 7.2.2

Let p(n) be a polynomial of degree k: $p(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0$ where $a_k > 0$ Then, p(n) is $\Theta(n^k)$

Asymptotic Growth Logarithmic Functions of Different Bases

• If a and b constants such a > 1 and b > 1, then $\log_a(n)$ is $\Theta(\log_b(n))$

• Proof:

1.
$$n = a^{\log_a(n)} = b^{\log_b(n)}$$
 when $n \ge 1$

2.
$$\log_a(a^{\log_a(n)}) = \log_a(b^{\log_b(n)})$$
 when $n \ge 1$

3. $\log_a(n) = \log_a(b) \cdot \log_b(n)$ where $\log_a(b)$ is positive and $n \ge 1$

4. $\log_a(n) = c \cdot \log_b(n)$ where $c = \log_a(b)$ is positive and $n \ge 1$

5.
$$\log_a(n) \le c \cdot \log_b(n)$$
 where *c* is positive and $n \ge 1$

6.
$$\log_a(n) \ge c \cdot \log_b(n)$$
 where *c* is positive and $n \ge 1$

7. $\log_a(n)$ is $\Theta(\log_b(n))$

Growth Rates of Common Functions

- A function is a <u>constant function</u> if it always returns the same value
- If f(n) is a constant function, then f(n) is $\Theta(1)$
- f(n) is called <u>linear</u> if f(n) is $\Theta(n)$
- f(n) is called <u>polynomial</u> if f(n) is $\Theta(n^k)$ for a real number k > 0
- f(n) is called <u>exponential</u> if f(n) is $\Theta(c^n)$ for a real number c > 1

Common Functions in Algorithmic Complexity

Function	Name
Θ(1)	Constant
$\Theta(\log(\log(n)))$	Log log
$\Theta(\log(n))$	Logarithmic
$\Theta(n)$	Linear
$\Theta(n \cdot \log(n))$	n log n
$\Theta(n^2)$	Quadratic
$\Theta(n^3)$	Cubic
$\Theta(n^k) k > 3$	Power
$\Theta(c^n) c > 1$	Exponential
$\Theta(n!)$	Factorial

Rules for Asymptotic Growth of Functions

- If f(n) is $\mathcal{O}(h(n))$ and g(n) is $\mathcal{O}(h(n))$, then f(n) + g(n) is $\mathcal{O}(h(n))$
- If f(n) is $\Omega(h(n))$ and g(n) is $\Omega(h(n))$, then f(n) + g(n) is $\Omega(h(n))$
- If f(n) is $\mathcal{O}(g(n))$ and a > 0, then $a \cdot f(n)$ is $\mathcal{O}(g(n))$
- If f(n) is $\Omega(g(n))$ and a > 0, then $a \cdot f(n)$ is $\Omega(g(n))$
- If f(n) is $\mathcal{O}(g(n))$ and g(n) is $\mathcal{O}(h(n))$, then f(n) is $\mathcal{O}(h(n))$
- If f(n) is $\Omega(g(n))$ and g(n) is $\Omega(h(n))$, then f(n) is $\Omega(h(n))$