CS 3333: Mathematical Foundations

Elementary Matrix Operations

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- If $|A| \neq 0$, then A is a **non-singular** matrix, and there exists an $n \times n$ matrix, denoted A^{-1}, such that $A \cdot A^{-1} = I_n$.
- A^{-1} is unique.

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- One can verify that $A \cdot A^{-1} = I_2$.
- It is more complicated to compute the inverse of larger square matrix.

Example

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $|A| = 1 * 4 - 3 * 2 = -2$, $A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

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- $A * A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} * \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$
- $A * A^{-1} = \begin{pmatrix} 1*(-2) + 2*\frac{3}{2} & 1*1 + 2*(-\frac{1}{2}) \\ 3*(-2) + 4*\frac{3}{2} & 3*1 + 4*(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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- $$\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}$$

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- We call this matrix an **augmented matrix**.
- We then use **elementary row operations** to reduce the left half of the augmented matrix to the identity matrix. The right half of the resulting augmented matrix is A^{-1}.

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- Why it works?
- $(A|I_n) = (AA^{-1}|I_nA^{-1}) = (I_n|A^{-1})$
- How to transform $(A|I_n)$ to $(I_n|A^{-1})$?

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- Consider the system of three equations in three unknowns
  - $x_1 + x_3 = 1$
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  - $x_2 + 2x_3 = 1$
- It can be written in matrix form as $Ax = b$ where
  - $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$
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  - $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 0 & 1 & 2 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $b = \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \\ 1 \end{pmatrix}$

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- Multiply with the constant 3 on the second equation. We get
  - $A = \begin{pmatrix} 1 & 0 & 1 \\ \mathbf{6} & \mathbf{3} & \mathbf{0} \\ 0 & 1 & 2 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ \mathbf{9} \\ 1 \end{pmatrix}$

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  - $A = \begin{pmatrix} 1 & 0 & 1 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} \\ 0 & 1 & 2 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ \mathbf{1} \\ 1 \end{pmatrix}$

Elementary Matrix Operations

- We have the following **Elementary Row Operations** in modifying a matrix.

1. Row switching
2. Row multiplication by a constant
3. Replace a row by a sum of that row and a multiple of another row.

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- We can define elementary column operations similarly.

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- If A is non-singular, then A^{-1} exists.
- $A^{-1}Ax = A^{-1}b \implies Ix = A^{-1}b \implies x = A^{-1}b.$
- Therefore, if we compute A^{-1}, we can solve the system of linear equations by computing $A^{-1}b$.

Elementary Matrix Operations

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -4 & 2 & 0 & 1 & 0 \\ 5 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3=r_3-5r_1]{r_2=r_2-3r_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -10 & -1 & -3 & 1 & 0 \\ 0 & -7 & 0 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2=-r_2/10} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/10 & 3/10 & -1/10 & 0 \\ 0 & -7 & 0 & -5 & 0 & 1 \end{array}\right] \xrightarrow[r_3=r_3+7r_2]{r_1=r_1-2r_2}$$

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$$\begin{array}{rrr|rrr} 1 & 0 & 4/5 & 2/5 & 1/5 & 0 \\ 0 & 1 & 1/10 & 3/10 & -1/10 & 0 \\ 0 & 0 & 7/10 & -29/10 & -7/10 & 1 \end{array} \xrightarrow{r_3 = (10/7) * r_3}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 26/7 & 1 & -8/7 \\ 0 & 1 & 0 & 5/7 & 0 & -1/7 \\ 0 & 0 & 1 & -29/7 & -1 & 10/7 \end{array}\right]$$

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$$Ax = b. \quad A^{-1}Ax = A^{-1}\cdot b$$

$$\Rightarrow x = A^{-1}\cdot b$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{26}{7} & 1 & -\frac{8}{7} \\ \frac{5}{7} & 0 & -\frac{1}{7} \\ -\frac{29}{7} & -1 & \frac{10}{7} \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

$$x_1 = \frac{26}{7}\cdot 4 + 1\cdot 2 + \frac{8}{7} = 18$$

$$x_2 = \frac{5}{7}\cdot 4 + 0\cdot 2 + \frac{1}{7} = 3$$

$$x_3 = -\frac{29}{7}\cdot 4 + -1\cdot 2 + \frac{-10}{7} = -20$$