# Section 8.5
# More Inductive Proofs

# Induction Hypothesis

- When proving $\forall n P(n)$ by mathematical induction, we prove $P(k) \rightarrow P(k+1)$ by assuming $P(k)$ and deriving $P(k+1)$
- $P(k)$ is called the <u>induction hypothesis</u>

# Proofs of Inequalities

- Example 5: Prove $\forall n\ P(n)$ by mathematical induction on the natural numbers where

$$P(n) \text{ is } n < 2^n$$

1. Base case: Prove $P(0)$

$$0 < 1 = 2^0$$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $k < 2^k$

   and $P(k+1)$ is $k+1 < 2^{k+1}$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $k < 2^k$ We assume this

   and $P(k+1)$ is $k+1 < 2^{k+1}$ We must conclude this

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\boxed{k} < 2^k$

   and $P(k+1)$ is $\boxed{k} + 1 < 2^{k+1}$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

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    1. Assume $k < 2^k$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \to P(k+1)$

    1. Assume $k < 2^k$
    2. $k + 1 < 2^k + 1$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   1. Assume $k < 2^k$
   2. $k + 1 < 2^k + 1$
   3. $\leq 2^k + 2^k$ Since $k \in \boldsymbol{N}$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k < 2^k$
    2. $k + 1 < 2^k + 1$
    3. $\leq 2^k + 2^k$ Since $k \in \boldsymbol{N}$
    4. $= 2 \cdot 2^k$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k < 2^k$
    2. $k + 1 < 2^k + 1$
    3. $\leq 2^k + 2^k$ Since $k \in \boldsymbol{N}$
    4. $= 2 \cdot 2^k$
    5. $= 2^{k+1}$

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

| | | |
|---|---|---|
| 1. | Assume $k < 2^k$ | |
| 2. | $k + 1 < 2^k + 1$ | |
| 3. | $\leq 2^k + 2^k$ | Since $k \in \boldsymbol{N}$ |
| 4. | $= 2 \cdot 2^k$ | |
| 5. | $= 2^{k+1}$ | |
| 6. | $k + 1 < 2^{k+1}$ | |

# Proofs of Inequalities

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

| | | |
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| 1. | Assume $k < 2^k$ | |
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| 3. | $\leq 2^k + 2^k$ | Since $k \in \boldsymbol{N}$ |
| 4. | $= 2 \cdot 2^k$ | |
| 5. | $= 2^{k+1}$ | |
| 6. | $k + 1 < 2^{k+1}$ | |

Lines 2-5: $k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

# Lower Bounds

- Example 6: Prove $\forall n P(n)$ by mathematical induction on the natural numbers where

$$P(n) \text{ is } 2^n < n! \text{ when } n \geq 4$$

Instead, use mathematical induction on the set $\{4, 5, 6, \ldots\}$.

For this set, the base case is $P(4)$ and the induction step is still $P(k) \rightarrow P(k+1)$

# Lower Bounds

- Example 6: Prove $\forall n P(n)$ by mathematical induction on the set $\{4, 5, 6, \ldots\}$ where

$$P(n) \text{ is } 2^n < n!$$

1. Base case: Prove $P(4)$

$$2^4 = 16 < 24 = 1 \cdot 2 \cdot 3 \cdot 4 = 4!$$

# Lower Bounds

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $2^k < k!$ — We assume this

   and $P(k+1)$ is $2^{k+1} < (k+1)!$ — We must conclude this

# Lower Bounds

2. Prove $P(k) \to P(k+1)$

   Note that $P(k)$ is $2^k < k!$

   and $P(k+1)$ is $2^{k+1} < (k+1)!$

# Lower Bounds

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $2^k < k!$

    -

# Lower Bounds

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $2^k < k!$
    2. $2^k \cdot 2 < k! \cdot 2$

# Lower Bounds

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $2^k < k!$
    2. $2^k \cdot 2 < k! \cdot 2$
    3. $2^{k+1} < k! \cdot 2$

# Lower Bounds

2. Prove $P(k) \to P(k+1)$

    1. Assume $2^k < k!$
    2. $2^k \cdot 2 < k! \cdot 2$
    3. $2^{k+1} < k! \cdot 2$
    4. $\quad\quad < k! \cdot (k+1)$ since $k \in \{4, 5, 6, \ldots\}$

# Lower Bounds

2. Prove $P(k) \rightarrow P(k+1)$

   1. Assume $2^k < k!$
   2. $2^k \cdot 2 < k! \cdot 2$
   3. $2^{k+1} < k! \cdot 2$
   4. $\quad < k! \cdot (k+1)$ since $k \in \{4, 5, 6, \dots\}$
   5. $\quad = (k+1)!$

# Lower Bounds

2. Prove $P(k) \rightarrow P(k+1)$

   1. Assume $2^k < k!$
   2. $2^k \cdot 2 < k! \cdot 2$
   3. $2^{k+1} < k! \cdot 2$
   4. $\quad < k! \cdot (k+1)$ since $k \in \{4, 5, 6, \ldots\}$
   5. $\quad = (k+1)!$
   6. $2^{k+1} < (k+1)!$

# Divisibility

- Example 8: Prove $\forall n\ P(n)$ by mathematical induction on the natural numbers where

$P(n)$ is $n^3 - n$ is divisible by 3

1. Base case: Prove $P(0)$

$$0^3 - 0 = 0 = 3 \cdot 0$$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $k^3 - k = 3i$ for some integer $i$

   and $P(k+1)$ is $(k+1)^3 - (k+1) = 3i$ for some integer $i$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $k^3 - k = 3i$ for some integer $i$ — We assume this

   and $P(k+1)$ is $(k+1)^3 - (k+1) = 3i$ for some integer $i$

   We must conclude this

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\boxed{k^3 - k} = 3i$ for some integr $i$

   and $P(k+1)$ is $(k+1)^3 - (k+1) = 3i$ for some integer $i$

# Multiplication Instead of Addition

$P(k+1)$ is $(k+1)^3-(k+1) = 3i$ for some integer $i$

$$(k+1)^3-(k+1)$$

# Multiplication Instead of Addition

$P(k+1)$ is $(k+1)^3-(k+1) = 3i$ for some integer $i$

$$(k+1)^3-(k+1) \; = (k^3+3k^2+3k+1)-(k+1)$$

# Multiplication Instead of Addition

$P(k+1)$ is $(k+1)^3-(k+1)=3i$ for some integer $i$

$$\begin{aligned}(k+1)^3-(k+1) &= (k^3+3k^2+3k+1)-(k+1)\\ &= k^3+3k^2+3k-k\end{aligned}$$

# Multiplication Instead of Addition

$P(k+1)$ is $(k+1)^3-(k+1) = 3i$ for some integer $i$

$$\begin{aligned}(k+1)^3-(k+1) &= (k^3+3k^2+3k+1)-(k+1)\\ &= k^3+3k^2+3k-k\\ &= \boxed{k^3-k}+3k^2+3k\end{aligned}$$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

   1. Assume $k^3 - k = 3i$ for some integer $i$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k^3 - k = 3i$ for some integer $i$
    2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k^3 - k = 3i$ for some integer $i$
    2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$
    3. $k^3 + 3k^2 + 3k - k = 3i + 3k^2 + 3k$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k^3 - k = 3i$ for some integer $i$
    2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$
    3. $k^3 + 3k^2 + 3k - k = 3i + 3k^2 + 3k$
    4. $k^3 + 3k^2 + 3k - k + 1 - 1 = 3i + 3k^2 + 3k$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k^3 - k = 3i$ for some integer $i$
    2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$
    3. $k^3 + 3k^2 + 3k - k = 3i + 3k^2 + 3k$
    4. $k^3 + 3k^2 + 3k - k + 1 - 1 = 3i + 3k^2 + 3k$
    5. $k^3 + 3k^2 + 3k + 1 - k - 1 = 3i + 3k^2 + 3k$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k^3 - k = 3i$ for some integer $i$
    2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$
    3. $k^3 + 3k^2 + 3k - k = 3i + 3k^2 + 3k$
    4. $k^3 + 3k^2 + 3k - k + 1 - 1 = 3i + 3k^2 + 3k$
    5. $k^3 + 3k^2 + 3k + 1 - k - 1 = 3i + 3k^2 + 3k$
    6. $k^3 + 3k^2 + 3k + 1 - (k+1) = 3i + 3k^2 + 3k$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

   1. Assume $k^3 - k = 3i$ for some integer $i$
   2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$
   3. $k^3 + 3k^2 + 3k - k = 3i + 3k^2 + 3k$
   4. $k^3 + 3k^2 + 3k - k + 1 - 1 = 3i + 3k^2 + 3k$
   5. $k^3 + 3k^2 + 3k + 1 - k - 1 = 3i + 3k^2 + 3k$
   6. $k^3 + 3k^2 + 3k + 1 - (k+1) = 3i + 3k^2 + 3k$
   7. $(k+1)^3 - (k+1) = 3i + 3k^2 + 3k$

# Multiplication Instead of Addition

2. Prove $P(k) \rightarrow P(k+1)$

    1. Assume $k^3 - k = 3i$ for some integer $i$
    2. $k^3 - k + 3k^2 + 3k = 3i + 3k^2 + 3k$
    3. $k^3 + 3k^2 + 3k - k = 3i + 3k^2 + 3k$
    4. $k^3 + 3k^2 + 3k - k + 1 - 1 = 3i + 3k^2 + 3k$
    5. $k^3 + 3k^2 + 3k + 1 - k - 1 = 3i + 3k^2 + 3k$
    6. $k^3 + 3k^2 + 3k + 1 - (k+1) = 3i + 3k^2 + 3k$
    7. $(k+1)^3 - (k+1) = 3i + 3k^2 + 3k$
    8. $(k+1)^3 - (k+1) = 3(i + k^2 + k)$

# Another Example

- Example 9: Prove $\forall n P(n)$ by mathematical induction where

$P(n)$ is $7^{n+2} + 8^{2n+1}$ is divisible by 57

1. Prove $P(0)$

$$7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 49 + 8 = 57$$

# Another Example

2. Prove $P(k) \rightarrow P(k+1)$

   We assume this

   Note that $P(k)$ is $7^{k+2} + 8^{2k+1}$ is divisible by 57

   and $P(k+1)$ is $7^{k+1+2} + 8^{2(k+1)+1}$ is divisible by 57

   We must conclude this

# Another Example

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $7^{k+2} + 8^{2k+1}$ is divisible by 57

   and $P(k+1)$ is $7^{k+1+2} + 8^{2(k+1)+1}$ is divisible by 57

# Another Example

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $7^{k+2} + 8^{2k+1}$ is divisible by 57

   and $P(k+1)$ is $7^{k+1+2} + 8^{2(k+1)+1}$ is divisible by 57

$$
\begin{aligned}
7^{k+1+2} + 8^{2(k+1)+1} &= 7^{k+1+2} + 8^{2(k+1)+1} \\
&= 7^{k+1+2} + 8^{2k+2+1} \\
&= 7 \cdot 7^{k+2} + 8^{2} \cdot 8^{2k+1} \\
&= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\
&= 7 \cdot 7^{k+2} + (7+57) \cdot 8^{2k+1} \\
&= 7 \cdot 7^{k+2} + 7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1} \\
&= 7\left(7^{k+2} + 8^{2k+1}\right) + 57 \cdot 8^{2k+1}
\end{aligned}
$$

# Another Example

2. Prove $P(k) \rightarrow P(k+1)$

| | | |
|---|---|---|
| $7^{k+2} + 8^{2k+1}$ | is divisible by 57 | Assumption |
| $7^{k+2} + 8^{2k+1}$ | $= 57i$ for some integer $i$ | |
| $7\left(7^{k+2} + 8^{2k+1}\right) + 57 \cdot 8^{2k+1}$ | $= 7 \cdot 57i + 57 \cdot 8^{2k+1}$ | |
| $7\left(7^{k+2} + 8^{2k+1}\right) + 57 \cdot 8^{2k+1}$ | $= 57\left(7i + 8^{2k+1}\right)$ | |
| $7 \cdot 7^{k+2} + 7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1}$ | $= 57\left(7i + 8^{2k+1}\right)$ | |
| $7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1}$ | $= 57\left(7i + 8^{2k+1}\right)$ | |
| $7^{k+2+1} + 8^{2k+2+1}$ | $= 57\left(7i + 8^{2k+1}\right)$ | |
| $7^{k+1+2} + 8^{2(k+1)+1}$ | $= 57\left(7i + 8^{2k+1}\right)$ | |
| $7^{k+1+2} + 8^{2(k+1)+1}$ | is divisible by 57 | |

# Induction on the Size of Sets

- Example 10: Prove $\forall n P(n)$ by mathematical induction on the natural numbers where

  $P(n)$ is: The power set of a set with $n$ elements has $2^n$ elements

  1. Prove $P(0)$

$$|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1 = 2^0$$

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is: A set with $k$ elements has $2^k$ subsets

   and $P(k+1)$ is: A set with $k+1$ elements has $2^{k+1}$ subsets

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

We assume this

Note that $P(k)$ is A set with $k$ elements has $2^k$ subsets

and $P(k+1)$ is A set with $k+1$ elements has $2^{k+1}$ subsets

We must conclude this

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is A set with $k$ elements has $2^k$ subsets

   and $P(k+1)$ is A set with $k+1$ elements has $2^{k+1}$ subsets

   We need to be able to count the subsets of a set. Given that a set with $k$ elements has $2^k$ subsets, how do we count the additional subsets that are possible by adding another element to the set?

# The Subsets of $\{a, b, c\}$ and $\{a, b, c\} \cup \{d\}$

| The subsets of $\{a, b, c\}$ | The subsets of $\{a, b, c\}$ combined with $d$ |
|---|---|
| $\emptyset$ | $\emptyset \cup \{d\}$ |
| $\{a\}$ | $\{a\} \cup \{d\}$ |
| $\{b\}$ | $\{b\} \cup \{d\}$ |
| $\{c\}$ | $\{c\} \cup \{d\}$ |
| $\{a, b\}$ | $\{a, b\} \cup \{d\}$ |
| $\{a, c\}$ | $\{a, c\} \cup \{d\}$ |
| $\{b, c\}$ | $\{b, c\} \cup \{d\}$ |
| $\{a, b, c\}$ | $\{a, b, c\} \cup \{d\}$ |
| The subsets of $\{a, b, c\} \cup \{d\}$ that do not contain $d$ | The subsets of $\{a, b, c\} \cup \{d\}$ that contain $d$ |

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Assume that a set with $k$ elements has $2^k$ subsets.

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Assume that a set with $k$ elements has $2^k$ subsets.

   Add a $k+1st$ element to the set to get a set with $k+1$ elements.

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Assume that a set with $k$ elements has $2^k$ subsets.

   Add a $k+1st$ element to the set to get a set with $k+1$ elements.

   Consider the subsets of the new set of $k+1$ elements. Each subset either does not contain the $k+1st$ element or does contain it.

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Assume that a set with $k$ elements has $2^k$ subsets.

   Add a $k+1st$ element to the set to get a set with $k+1$ elements.

   Consider the subsets of the new set of $k+1$ elements. Each subset either does not contain the $k+1st$ element or does contain it.

   - The subsets that do not contain the $k+1st$ element are also the subsets of the set with $k$ elements. By the assumption, there are $2^k$ such subsets.

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Assume that a set with $k$ elements has $2^k$ subsets.

   Add a $k+1st$ element to the set to get a set with $k+1$ elements.

   Consider the subsets of the new set of $k+1$ elements. Each subset either does not contain the $k+1st$ element or does contain it.

   - The subsets that do not contain the $k+1st$ element are also the subsets of the set with $k$ elements. By the assumption, there are $2^k$ such subsets.
   - The subsets that do contain the $k+1st$ element are the result of adding the $k+1st$ element to each of the $2^k$ such subsets of the original set. By the assumption there are $2^k$ such subsets.

# Induction on the Size of Sets

2. Prove $P(k) \rightarrow P(k+1)$

   Assume that a set with $k$ elements has $2^k$ subsets.

   Add a $k+1st$ element to the set to get a set with $k+1$ elements.

   Consider the subsets of the new set of $k+1$ elements. Each subset either does not contain the $k+1st$ element or does contain it.

   - The subsets that do not contain the $k+1st$ element are also the subsets of the set with $k$ elements. By the assumption, there are $2^k$ such subsets.
   - The subsets that do contain the $k+1st$ element are the result of adding the $k+1st$ element to each of the $2^k$ such subsets of the original set. By the assumption there are $2^k$ such subsets.
   - Thus, there are a total of $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets