Homework Assignment 5
CS 2233
Section 001 and 002
Due: 11:59pm Friday, March 8

Problem 1. [20 points]
Complete all participation activities in zyBook sections 4.5, 7.1-7.3

Problem 2. [10 points]
Find $f \circ g$ and $g \circ f$ where $f, g: \boldsymbol{R} \to \boldsymbol{R}$ with $f(x) = 3x + 4$ and $g(x) = x^2$

$(f \circ g)(x) = f\big(g(x)\big) = 3g(x) + 4 = 3x^2 + 4$

$(g \circ f)(x) = g\big(f(x)\big) = (3x + 4)^2 = 9x^2 + 24x + 16$

Problem 3. [20 points]
Determine whether each of the following functions is $\mathcal{O}(x^2)$. If a function is $\mathcal{O}(x^2)$, then prove it by deriving witnesses c and n_0.
a. [5 points] $100x + 1000$

1.	$100x$	$\leq x^2$	when $x \geq 100$
2.	1000	$\leq x^2$	when $x \geq 100$
3.	$100x + 1000$	$\leq 2x^2$	when $x \geq 100$

$100x + 1000$ is $\mathcal{O}(x^2)$ with witnesses $c = 2$ and $n_0 = 100$

b. [5 points] $100x^2 + 1000$

1.	$100x^2$	$\leq 100x^2$	
2.	1000	$\leq x^2$	when $x \geq 100$
3.	$100x^2 + 1000$	$\leq 101x^2$	when $x \geq 100$

$100x^2 + 1000$ is $\mathcal{O}(x^2)$ with witnesses $c = 101$ and $n_0 = 100$

c. [5 points]
$\frac{x^3}{100} - 1000x^2$ is not $\mathcal{O}(x^2)$

d. [5 points] $x \cdot \log(x)$

1.	x	$\leq x$	
2.	$\log(x)$	$\leq x$	when $x \geq 1$
3.	$x \cdot \log(x)$	$\leq x^2$	when $x \geq 1$

$x \cdot \log(x)$ is $\mathcal{O}(x^2)$ with witnesses $c = 1$ and $n_0 = 1$

Problem 4. [10 points]
a. [5 points] Use the definition of Big-Θ to show that $5n^5 + 4n^4 + 3n^3 + n$ is $\Theta(n^5)$

Let $n \geq 1$, then

1.	$5n^5$	$\leq 5n^5$	
2.	$4n^4$	$\leq 4n^5$	when $n \geq 1$
3.	$3n^3$	$\leq 3n^5$	when $n \geq 1$
4.	n	$\leq n^5$	when $n \geq 1$

5. $5n^5 + 4n^4 + 3n^3 + n \quad \leq 13n^5$ when $n \geq 1$

So $5n^5 + 4n^4 + 3n^3 + n$ is $\mathcal{O}(n^5)$ with witnesses $c = 13$ and $n_0 = 1$

In addition, when $n \geq 1$, $5n^5 + 4n^4 + 3n^3 + n \geq n^5$, so $5n^5 + 4n^4 + 3n^3 + n$ is $\Omega(n^5)$ with witnesses $c = 1$ and $n_0 = 1$

Hence $5n^5 + 4n^4 + 3n^3 + n$ is $\Theta(n^5)$

b. [5 points] Use the definition of Big-Θ to show that $2n^3 - n + 10$ is $\Theta(n^3)$

Let $n \geq 1$, then

1. $2n^3 \leq 2n^3$
2. $-n \leq 0$ when $n \geq 0$
3. $10 \leq n^3$ when $n \geq 3$
4. $2n^3 - n + 10 \leq 3n^3$ when $n \geq 0$ and $n \geq 3$
5. $2n^3 - n + 10 \leq 3n^3$ when $n \geq 3$

So $2n^3 - n + 10$ is $\mathcal{O}(n^3)$ with witnesses $c = 3$ and $n_0 = 3$

In addition,

1. $n^3 + 10 \geq n^3$
2. $n^3 \geq n$ when $n \geq 1$
3. $2n^3 + 10 \geq n^3 + n$ when $n \geq 1$
4. $2n^3 - n + 10 \geq n^3$ when $n \geq 1$

So $2n^3 - n + 10$ is $\Omega(n^3)$ with witnesses $c = 1$ and $n_0 = 1$
Thus $2n^3 - n + 10$ is $\Theta(n^3)$

Problem 5. [10 points] Prove each of the following by deriving witnesses c and n_0.
a. [5 points] If $f(n)$ is $\mathcal{O}(g(n))$ and $a > 0$, then $a \cdot f(n)$ is $\mathcal{O}(g(n))$

1. Assume $f(n)$ is $\mathcal{O}(g(n))$ and $a > 0$
2. $f(n) \leq c_f \cdot g(n)$ when $n \geq n_f$ for some positive c_f and n_f
3. $a \cdot f(n) \leq a \cdot c_f \cdot g(n)$ when $n \geq n_f$
4. $a \cdot f(n)$ is $\mathcal{O}(g(n))$ with witnesses positive $c = a \cdot c_f$ and $n_0 = n_f$
5. If $f(n)$ is $\mathcal{O}(g(n))$ and $a > 0$, then $a \cdot f(n)$ is $\mathcal{O}(g(n))$

b. [5 points] If $f(n)$ is $\Omega(g(n))$ and $g(n)$ is $\Omega(h(n))$, then $f(n)$ is $\Omega(h(n))$

1. Assume $f(n)$ is $\Omega(g(n))$ and $g(n)$ is $\Omega(h(n))$
2. $f(n) \geq c_f \cdot g(n)$ when $n \geq n_f$ for some positive c_f and n_f
3. $g(n) \geq c_g \cdot h(n)$ when $n \geq n_g$ for some positive c_g and n_g
4. $f(n) \geq c_f \cdot c_g \cdot h(n)$ when $n \geq n_f$ and $n \geq n_g$
5. $f(n) \geq c_f \cdot c_g \cdot h(n)$ when $n \geq \max(n_f, n_g)$
6. $f(n)$ is $\Omega(h(n))$ with witnesses $c = c_f \cdot c_g$ and $n_0 = \max(n_f, n_g)$
7. If $f(n)$ is $\Omega(g(n))$ and $g(n)$ is $\Omega(h(n))$, then $f(n)$ is $\Omega(h(n))$