Homework Assignment 6
CS 2233
Sections 001 and 002
Due: 11:59pm Friday, March 22

Problem 1. [10 points]
Complete all participation activities in zyBook sections 8.1-8.5

Problem 2. [10 points]
What are the first 4 terms of the following sequences:

a. [5 points] $\{(-3)^n\}_{n\in \boldsymbol{N}}$

$1, -3, 9, -27$

b. [5 points] $\{(-1)^n + 1\}_{n\in \boldsymbol{N}}$

$2, 0, 2, 0$

Problem 3. [10 points]
Use index substitution to rewrite the following summation so that the index starts at 0. Then use the closed form of the geometric series to compute the value of the summation:

$$\sum_{i=3}^{6}(-2)^{i-3}$$

$$\begin{aligned}
\sum_{i=3}^{6}(-2)^{i-3} &= (-2)^{3-3} + (-2)^{4-3} + (-2)^{5-3} + (-2)^{6-3} \\
&= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 \\
&= \sum_{i=0}^{3}(-2)^i \\
&= \frac{1(-2)^{3+1} - 1}{-2 - 1} \\
&= \frac{(-2)^4 - 1}{-3} \\
&= -5
\end{aligned}$$

Problem 4. [15 points]
Express each sequence below using compact notation.

a. [5 points] $4, 10, 16, 22, 28, 34, 40, \ldots$

$\{4 + 6n\}_{n\in \boldsymbol{N}}$

b. [5 points] $5, 15, 45, 135, 405, \ldots$

$\{5 \cdot 3^n\}_{n\in \boldsymbol{N}}$

c. [5 points] $10, 20, 10, 20, 10, 20, 10, \ldots$

Possible Solutions:

$\{15 - 5(-1)^n\}_{n\in \boldsymbol{N}}$

$\{15 + 5(-1)^{n+1}\}_{n \in \boldsymbol{N}}$
$\{15 + 5(-1)^{n}\}_{n \in \boldsymbol{Z}^+}$

Problem 5. [20 points] Consider a proof by mathematical induction on the positive integers of $P(n)$ where $P(n)$ is: The sum of the first n even positive integers is $n^2 + n$.

a. [5 points] What is the statement $P(1)$?

$2 = 1^2 + 1$

b. [5 points] What is the induction hypothesis?

The sum of the first k even positive integers is $k^2 + k$

c. [5 points] What do you need to prove in the inductive step?

If $P(k)$ then $P(k+1)$, or: If the sum of the first k even positive integers is $k^2 + k$, then the sum of the first $k+1$ even positive integers is $(k+1)^2 + (k+1)$

d. [5 points] Complete the proof of the inductive step.

1. Assume the sum of the first k even positive integers is $k^2 + k$
2. $(2 \cdot 1) + (2 \cdot 2) + \cdots + 2 \cdot k \quad = k^2 + k$
3. $(2 \cdot 1) + (2 \cdot 2) + \cdots + 2 \cdot k + 2 \cdot (k+1) \quad = k^2 + k + 2 \cdot (k+1)$
4. $= k^2 + k + 2k + 2$
5. $= (k^2 + 2k + 1) + (k+1)$
6. $= (k+1)^2 + (k+1)$
7. The sum of the first $k+1$ even positive integers is $(k+1)^2 + (k+1)$

Problem 6. [5 points] Use mathematical induction to prove $3^n < n!$ for all integers $n > 6$.

1. Base case: $n = 7$
$3^7 = 2187 < 5040 = 7!$

2. Induction: Prove if $3^k < k!$ then $3^{k+1} < (k+1)!$

1. Assume $3^k \quad < k!$
2. $3^{k+1} \quad < 3k!$
3. $< (k+1)k! \quad$ Since $k > 6$
4. $= (k+1)!$
5. $3^{k+1} \quad < (k+1)!$

Problem 7. [5 points] Use mathematical induction to prove $\log(n!) \leq n\log(n)$ for all integers $n \geq 1$.

Note that:
1) $\log(1) = 0$
2) $\log(a \cdot b) = \log(a) + \log(b)$
3) If $a \leq b$ then $\log(a) \leq \log(b)$

1. Base case: $n = 1$
$\log(1!) = 0 \leq 1 \cdot 0 = 1 \cdot \log(1)$

2. Induction: Prove if $\log(k!) \leq k\log(k)$ then $\log((k+1)!) \leq (k+1)\log(k+1)$

1. Assume $\log(k!) \quad \leq k\log(k)$

2.	$\log(k!) + \log(k+1)$	$\leq k\log(k) + \log(k+1)$
3.	$\log(k!\,(k+1))$	$\leq k\log(k) + \log(k+1)$
4.	$\log\left((k+1)!\right)$	$\leq k\log(k) + \log(k+1)$
5.		$\leq k\log(k+1) + \log(k+1)$
6.		$\leq (k+1)\log(k+1)$

Problem 8. [5 points] Prove by mathematical induction on the positive integers $\forall n P(n)$ where $P(n)$ is: If $A_1, A_2, \cdots, A_n$ and B are sets, then $(\bigcap_{i=1}^{n} A_i) \cup B = \bigcap_{i=1}^{n}(A_i \cup B)$.

Note:
1. $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$
2. $\bigcap_{i=1}^{n}(A_i \cup B) = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B)$

Hints:
1. For each $k \geq 1$, $\bigcap_{i=1}^{k+1} A_i = \left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}$
2. When X, Y, and Z are sets, $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$

1. Base case: $n = 1$

$\left(\bigcap_{i=1}^{1} A_i\right) \cup B = A_1 \cup B = \bigcap_{i=1}^{1}(A_i \cup B)$

Induction: Prove if then $\left(\bigcap_{i=1}^{k} A_i\right) \cup B = \bigcap_{i=1}^{k}(A_i \cup B)$ then then $\left(\bigcap_{i=1}^{k+1} A_i\right) \cup B = \bigcap_{i=1}^{k+1}(A_i \cup B)$

1.	Assume $\left(\bigcap_{i=1}^{k} A_i\right) \cup B$	$= \bigcap_{i=1}^{k}(A_i \cup B)$	
2.	$\left(\left(\bigcap_{i=1}^{k} A_i\right) \cup B\right) \cap (A_{k+1} \cup B)$	$= \left(\bigcap_{i=1}^{k}(A_i \cup B)\right) \cap (A_{k+1} \cup B)$	
3.	$\left(\left(\bigcap_{i=1}^{k} A_i\right) \cup B\right) \cap (A_{k+1} \cup B)$	$= \bigcap_{i=1}^{k+1}(A_i \cup B)$	Hint 1
4.	$\left(\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right) \cup B$	$= \bigcap_{i=1}^{k+1}(A_i \cup B)$	Hint 2
5.	$\left(\bigcap_{i=1}^{k+1} A_i\right) \cup B$	$= \bigcap_{i=1}^{k+1}(A_i \cup B)$	Hint 1