Important Logical Equivalences

Logical Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg\neg p \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Complement laws
$p \rightarrow q \equiv \neg p \vee q$ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Conditional identities

- **T** stands for any tautology such as $p \vee \neg p$
- **F** stands for any contradiction such as $p \wedge \neg p$
- These equivalences can be used as templates. We can substitute a proposition for all occurrences of a variable. E.g.: (substitute $(q \rightarrow r)$ for p in the Identity law)

$$(q \rightarrow r) \wedge \mathrm{T} \equiv (q \rightarrow r)$$

Working with Logical Equivalences

- Note that if A, B, and C are any propositions then:
  - $A \equiv B$ if and only if $B \equiv A$
  - If $A \equiv B$ and $B \equiv C$, then $A \equiv C$
  - If $A \equiv B$ then A and B can be substituted for each other in other propositions

- We can use the above properties to derive new logical equivalences from existing ones (in a manner similar to algebra)

Deriving New Logical Equivalences

We can write derivations vertically and include the names of the equivalences that we are using (Example: $p \to q \equiv \neg q \to \neg p$)

$p \to q$	$\equiv \neg p \lor q$	Conditional Identity
	$\equiv q \lor \neg p$	Commutative law
	$\equiv \neg\neg q \lor \neg p$	Double Negation law
	$\equiv \neg q \to \neg p$	Conditional Identity

Deriving New Logical Equivalences

Another example: $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$(p \rightarrow r) \vee (q \rightarrow r)$	$\equiv (\neg p \vee r) \vee (q \rightarrow r)$	Conditional Identity
	$\equiv (\neg p \vee r) \vee (\neg q \vee r)$	Conditional Identity
	$\equiv \neg p \vee \big(r \vee (\neg q \vee r)\big)$	Associative law
	$\equiv \neg p \vee \big((\neg q \vee r) \vee r\big)$	Commutative law
	$\equiv \neg p \vee \big(\neg q \vee (r \vee r)\big)$	Associative law
	$\equiv (\neg p \vee \neg q) \vee (r \vee r)$	Associative law
	$\equiv (\neg p \vee \neg q) \vee r$	Idempotent law
	$\equiv \neg(p \wedge q) \vee r$	De Morgan's law
	$\equiv (p \wedge q) \rightarrow r$	Conditional Identity

Note how we get rid of implications in order to work with disjunctions

Deriving New Logical Equivalences

Yet another example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology. I.e., $(p \wedge q) \rightarrow (p \vee q) \equiv \mathrm{T}$

$(p \wedge q) \rightarrow (p \vee q)$	$\equiv \neg(p \wedge q) \vee (p \vee q)$	Conditional Identity
	$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	De Morgan
	$\equiv \neg p \vee \big(\neg q \vee (p \vee q)\big)$	Associative law
	$\equiv \neg p \vee \big(\neg q \vee (q \vee p)\big)$	Commutative law
	$\equiv \neg p \vee \big((\neg q \vee q) \vee p\big)$	Associative law
	$\equiv \neg p \vee \big(p \vee (\neg q \vee q)\big)$	Commutative law
	$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	Associative law
	$\equiv (\neg p \vee p) \vee (q \vee \neg q)$	Commutative law
	$\equiv (\neg p \vee p) \vee \mathrm{T}$	Complement law
	$\equiv \mathrm{T}$	Domination law

Note how we get rid of implications in order to work with disjunctions

Limitations of Propositional Logic

- Propositional logic is limited in how we can reason about statements
- For example:
  - Let p represent “Socrates is human"
  - Let q represent “All humans are mortal"
  - Let r represent “Socrates is mortal"
  - We cannot show that $p \land q \to r \equiv \mathrm{T}$ using propositional logic

Predicates

- The statement “Socrates is human” is a statement about Socrates, but it can be generalized to be applied to anyone:

  _______ is human

- We can do the same for “Socrates is mortal”:

  _______ is mortal

- Such generalizations are called predicates

Predicates

- Predicates can be about more than one thing.
- For example, from “Smith is taller than Jones”

_______ is taller than _______

- It is also possible to have predicates such as

_______ is taller than Jones

Smith is taller than _______

Predicate Logic

- Predicate logic uses capital letters ($P, Q, R, \ldots$) to represent predicates
- For example: Let H and M represent the predicates for being human and being mortal.
- If s stands for Socrates, then the meanings of $H(s)$ and $M(s)$ are:
  - "Socrates is human"
  - "Socrates is mortal"

Reintroducing the Logical Connectives

- The logical connectives from propositional logic: negation, conjunction, disjunction, implication and bi-implication are used in predicate logic also.

- Examples
  - $\neg P(x)$
  - $Q(x, y) \rightarrow P(x) \vee R(y)$

Variables and the Domain of Discourse

- In predicate logic, a domain of discourse (or universe of discourse) is the set of objects to which predicates may refer

- Predicates may take variables that have as values objects from the universe of discourse

Common Predicates

- In predicate logic, we often want to talk about two things being equal.
- We could explicitly state that a predicate P is the equality relation, i.e. $P(x, y)$ is true exactly when $x = y$.
- Instead, for convenience, we can use $x = y$ as a predicate.
- When the universe of discourse is numbers we can also use:
  - $x < y$, $x > y$, $x \leq y$, and $x \geq y$
  - the constants 0, 1, 2, … and the arithmetic operators +, -, etc.

Variables and the Universe of Discourse

- Example: Let S be a two-argument predicate symbol with the following meaning:
  - $S(x, y)$: x is smaller than y

- If the universe of discourse is the set of dogs at a dog park, then $\mathrm{S}(x, y)$ represents the statement that dog x is smaller than dog y

Examples

- Let the universe of discourse be a set of dogs at a dog park, $S(x, y)$ represent the statement x is smaller than y, and $B(x)$ represent the statement that x is a beagle

- Translate the following into predicate logic:
  - x is smaller than both y and z
  - y and z are the same size
  - If z is a beagle, then w is bigger than z

Examples

- x is smaller than both y and z
- $S(x, y) \wedge S(x, z)$

Examples

- y and z are the same size
- $\neg S(y, z) \wedge \neg S(z, y)$

Examples

- If z is a beagle, then w is bigger than z
- $B(z) \to S(z, w)$

Meaning

- To determine if $S(x, y)$ is true or false, we need to know which dogs x and y stand for.

- An environment, η, is a function that takes a variable and returns an object from the domain of discourse.

Meaning

- Given an environment, η, $S(x, y)$ is true exactly when the dog $\eta(x)$ is smaller than the dog $\eta(y)$

- The notation $[\![S(x, y)]\!]_{\eta}$ stands for the truth value of $S(x, y)$ given η

Logical Connectives and Environments

- The meaning of the connectives is the same as in propositional logic, but is expressed in terms of an environment function, η
- For example:

$[\![P(x) \wedge Q(y)]\!]_\eta$ is true exactly when $[\![P(x)]\!]_\eta$ and $[\![Q(y)]\!]_\eta$ are true

$[\![P(x) \vee Q(y)]\!]_\eta$ is true exactly when $[\![P(x)]\!]_\eta$ or $[\![Q(y)]\!]_\eta$ is true

$[\![P(x) \rightarrow Q(y)]\!]_\eta$ is true if $[\![P(x)]\!]_\eta$ being true implies $[\![Q(y)]\!]_\eta$ is true

Examples

- Let η be an environment function where:
  - $\eta(x) = 0$
  - $\eta(y) = 1$
  - $\eta(z) = 2$

- Then:

  $[\![y = x + y]\!]_\eta$ is true

  $[\![x > y \vee x > z]\!]_\eta$ is false

  $[\![\neg(x = 0 \rightarrow y = z)]\!]_\eta$ is true

Updating Environments

- An environment function η can be updated by altering its behavior on an input variable.
- Let D be the domain of discourse and also let d be a member of D
- $\eta_{x=d}$ is an environment that behaves exactly like η with the possible exception that $\eta_{x=d}(x) = d$

$$\eta_{x=d}(y) = \begin{cases} d & \text{if } y \text{ is } x \\ \eta(y) & \text{if } y \text{ is not } x \end{cases}$$

Updating Environments Example

- Assume the integers as a domain of discourse and the environment function η where:
  - $\eta(x) = 0$
  - $\eta(y) = 1$
  - $\eta(z) = 2$
- Then:
  - $\eta_{z=9}(x) = 0$
  - $\eta_{z=9}(y) = 1$
  - $\eta_{z=9}(z) = 9$

The Existential Quantifier

- Assume that the domain of discourse is the set of all dogs and that the predicate $S(x, y)$ is true exactly when x is smaller than y.
- How can we express that x is not the smallest dog?
- We would like to say that there exists a dog z such that $S(z, x)$

The Existential Quantifier

- To do this, predicate logic has an existential quantifier: $\exists$

$$\exists z\ S(z, x)$$

- We read this as "There exists a z such that $S(z, x)$"

- $[\![\exists z\ S(z, x)]\!]_\eta$ is true exactly when there is a d in the universe of discourse such that $[\![S(z, x)]\!]_{\eta_{z=d}}$ is true

Existential Quantifier Examples

- Translate the following into predicate logic:
  - x is not the biggest dog
  - x is the smallest dog
  - There is a beagle that is smaller than x
  - x is a beagle and there is another beagle that is the same size as x

Existential Quantifier Examples

- x is not the biggest dog

- $\exists z\, S(x, z)$

Existential Quantifier Examples

- x is the smallest dog

- $\neg \exists z \; S(z, x)$

Existential Quantifier Examples

- There is a beagle that is smaller than x
- $\exists y(B(y) \land S(y, x))$

Existential Quantifier Examples

- z is a beagle and there is another beagle that is the same size as z
- $B(z) \wedge \exists w(B(w) \wedge \neg(z = w) \wedge \neg S(z, w) \wedge \neg S(w, z))$

The Universal Quantifier

- Recall that we can express that x is the smallest dog in predicate logic as:

$$\neg\exists z \, S(z, x)$$

"It is not the case that there is a dog that is smaller than x"

- This could also be stated as:

"For any dog, z, z is not smaller than x "

The Universal Quantifier

- The statement

"For any dog, z, z is not smaller than x "

can be expressed as:

$$\forall z \neg S(z, x)$$

$[\![\forall z \neg S(z, x)\ ]\!]_{\eta}$ is true exactly when for each d in the universe of discourse $[\![\neg S(z, x)]\!]_{\eta_{z=d}}$ is true

Universal Quantifier Examples

- Translate the following into predicate logic:
  - All dogs are beagles
  - x is smaller than any beagle
  - x is a beagle and is smaller than all other beagles

Universal Quantifier Examples

- All dogs are beagles
- $\forall x \; B(x)$

Universal Quantifier Examples

- x is smaller than any beagle
- $\forall y\ (B(y) \to S(x, y))$

Universal Quantifier Examples

- x is a beagle and is smaller than all other beagles
- $B(x) \wedge \forall z \left(\left(B(z) \wedge \neg (x = z) \right) \rightarrow S(x, z) \right)$

More Examples

- Let the domain of discourse be the set of all people.
- Let $P(x, y)$ represent: x is the parent of y
- Let $L(x, y)$ represent: x loves y

- Translate the following to predicate logic:
  - u is a grandparent of v
  - x loves all of his/her children
  - y and z are siblings

More Examples

- u is a grandparent of v

- $\exists z\ (P(u, z) \wedge P(z, v)\ )$

More Examples

- x loves all of his/her children
- $\forall u\ (P(x,u) \rightarrow L(x,u)\ )$

More Examples

- y and z are siblings

- $\exists w(P(w, y) \land P(w, z)) \land \neg(y = z)$