Section 8.9
Structural Induction

Structural Induction

- A recursively defined set provides a pattern for proving properties about it
  1. Base case: Prove the property for the base elements of the set
  2. Induction step: Prove that if the property holds for elements that are used to construct a new element of the set, then the property is true for the new element

Structural Induction

- Example: Recall the definition of the set S:
  1. $3 \in S$ (3 is a base element)
  2. If $x \in S$ and $y \in S$, then $x + y \in S$
  3. Nothing else is in S

Prove by structural induction $\forall n P(n)$ where $P(n)$ is:

If $n \in S$, then n is a multiple of 3

Structural Induction

Example: Prove by structural induction $\forall n P(n)$ where $P(n)$ is:

If $n \in S$, then n is a multiple of 3

1. Base case: $3 \in S$

   3 is a multiple of 3

Structural Induction

Example: Prove by structural induction $\forall n P(n)$ where $P(n)$ is:

If $n \in S$, then n is a multiple of 3

2. Induction step: Assume $x \in S$ and $y \in S$ and that $P(x)$ and $P(y)$. Prove $P(x + y)$

Structural Induction

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2. Induction step: Assume $x \in S$ and $y \in S$ and that $P(x)$ and $P(y)$. Prove $P(x + y)$
    1. Assume $x \in S$, $y \in S$, and x and y are multiples of 3

Structural Induction

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   1. Assume $x \in S$, $y \in S$, and x and y are multiples of 3
   2. $x = 3i$ and $y = 3j$ for integers i and j

Structural Induction

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    1. Assume $x \in S$, $y \in S$, and x and y are multiples of 3
    2. $x = 3i$ and $y = 3j$ for integers i and j
    3. $x + y = 3i + 3j = 3(i + j)$

Structural Induction

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   1. Assume $x \in S$, $y \in S$, and x and y are multiples of 3
   2. $x = 3i$ and $y = 3j$ for integers i and j
   3. $x + y = 3i + 3j = 3(i + j)$
   4. $x + y$ is a multiple of 3

Well-Formed Formulas in Propositional Logic

- Let V be the set of propositional variables
- W, the set of well-formed formulas of propositional logic can be recursively defined as follows
  1. If $p \in V$ then $p \in W$
  2. If $w_1 \in W$ and $w_2 \in W$, then so are the following:
     - $(\neg w_1)$,
     - $(w_1 \wedge w_2)$
     - $(w_1 \vee w_2)$
     - $(w_1 \rightarrow w_2)$
     - $(w_1 \leftrightarrow w_2)$

Structural Induction

- Example: Prove by structural induction $\forall w\ P(w)$ where $P(w)$ is:

  If $w \in W$ then the w has an equal number of left and right parentheses

  1. Base case: $p \in W$ where p is a propositional variable

     Propositional variables have 0 left and right parentheses

Structural Induction

2. Induction step: Assume $w_1 \in W$ and $w_2 \in W$ and $P(w_1)$ and $P(w_2)$. Prove $P((\neg w_1))$, $P((w_1 \land w_2))$, $P((w_1 \lor w_2))$, $P((w_1 \to w_2))$, $P((w_1 \leftrightarrow w_2))$,
    1. Assume $w_1 \in W$ and $w_2 \in W$ and w_1 and w_2 each have an equal number of left and right parentheses
    2. There are 5 ways to use w_1 and w_2 to create a new formula
    3. Case 1: $(\neg w_1)$
    4. w_1 has the same number of left parentheses as right parentheses
    5. $(\neg w_1)$ has the same number of left parentheses as right parentheses

Structural Induction

2. Induction step continued

    6. Cases 2-5: Without loss of generality, consider $(w_1 \wedge w_2)$

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   8. $w_1 \wedge w_2$ has the same number of left and right parentheses

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   9. $(w_1 \wedge w_2)$ has the same number of left and right parentheses

Structural Induction

2. Induction step continued

6. Cases 2-5: Without loss of generality, consider $(w_1 \wedge w_2)$
7. w_1 and w_2 each have the same number of left and right parentheses
8. $w_1 \wedge w_2$ has the same number of left and right parentheses
9. $(w_1 \wedge w_2)$ has the same number of left and right parentheses
10. In all cases, the new formulas have the same number of left and right parentheses

Another Inductively Defined Set Proof

Define the set S as follows:

1. $1 \in S$
2. $3 \in S$
3. If $x \in S$ then $x + 4 \in S$

Another Inductively Defined Set Proof

- Prove by structural induction that each element of S is odd

1. Base cases: 1 and 3

   1 is odd and 3 is odd

Another Inductively Defined Set Proof

- Prove by structural induction that each element of S is odd

2. Induction step:
    1. Assume $x \in S$ and x is odd
    2. $x = 2i + 1$ for some integer i
    3. $x + 4 = 2i + 1 + 4$
    4. $x + 4 = 2(i + 2) + 1$
    5. $x + 4$ is odd

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