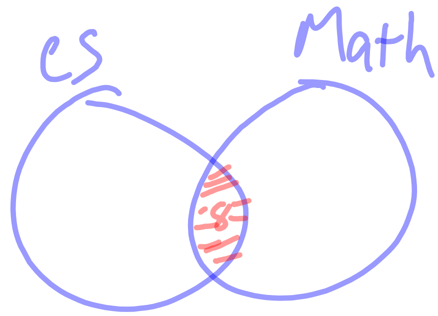
Generalized PIE Principle

Suppose that in a discrete math class, every student is a math or CS major. Suppose that 25 are CS majors 13 are math majors (with 8 students double majoring in math and CS). How many students are in the class?



A = set of all CS majors

B = set of all Math majors

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 25 + 13 - 8 = 30$$

P_1 = Students who are CS majors.

P_2 = Students who are Math majors.

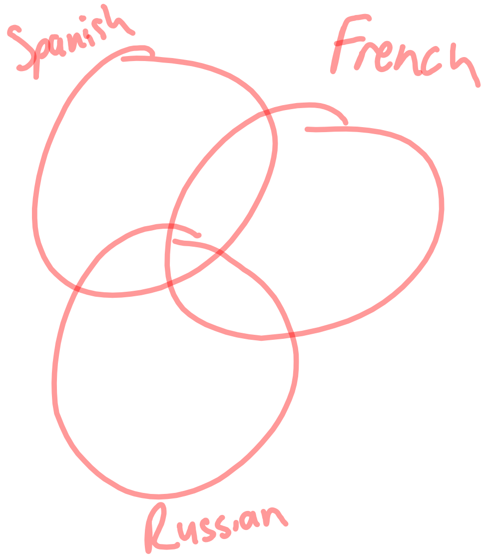
$N(P_1)$ = # of students who are CS majors

$N(P_2)$ = # of Math majors

$N(P_1 P_2)$ = # of students satisfying P_1 + P_2

Total # of students = $N(P_1) + N(P_2) - N(P_1 P_2)$.

Suppose a school offers Spanish, French, and Russian as foreign language classes.



P_1 = students taking Spanish

P_2 = " " French

P_3 = " " Russian

$N(P_1) = 1232, \quad N(P_2) = 879, \quad N(P_3) = 114$

$N(P_1P_2) = 103, \quad N(P_1P_3) = 23, \quad N(P_2,P_3) = 14$

$N(P_1P_2P_3) = 7$

Total # of students $= N(P_1) + N(P_2) + N(P_3) - N(P_1P_2) - N(P_1P_3) - N(P_2P_3) + N(P_1P_2P_3)$

$= 1232 + 879 + 114 - 103 - 23 - 14 + 7$

$= 2092$

Generalized Principle of Inclusion and Exclusion (PIE):

n sets: $A_1, A_2, \ldots, A_n$

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots \cup A_n|$$

$$= |A_1| + |A_2| + |A_3| + \cdots + |A_n| \quad \Leftarrow \binom{n}{1} \text{ terms}$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \quad \Leftarrow \binom{n}{2} \text{ terms}$$

$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \quad \binom{n}{3} \text{ terms}$$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4| - \cdots - |A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n|$$

$$\vdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n|$$

Example from Section 8.6 page 586: What is the number of primes between 2 and 100?

99 total numbers. We can count the number of composites and subtract this from 99

If a number is composite, there is a prime number that divides it.

Possible prime divisors are all primes between 2 and $\lfloor \sqrt{100} \rfloor$

2, 3, 5, 7

P_2 = # divisible by 2, P_3 = # divisible by 3

P_5 = # " " 5, P_7 = " " 7

Total # of primes: $99 - |P_2 \cup P_3 \cup P_5 \cup P_7| + 4$

$N(P_2) = \lfloor \frac{100}{2} \rfloor = 50$ $N(P_3) = \lfloor \frac{100}{3} \rfloor = 33$

$N(P_5) = \lfloor \frac{100}{5} \rfloor = 20$ $N(P_7) = \lfloor \frac{100}{7} \rfloor = 14$

$N(P_2P_3) = \lfloor \frac{100}{\text{lcm}(2,3)} \rfloor = \lfloor \frac{100}{6} \rfloor = 16$ $N(P_2P_5) = \lfloor \frac{100}{10} \rfloor = 10$

$N(P_2P_7) = \lfloor \frac{100}{14} \rfloor = 7$ $N(P_3P_5) = \lfloor \frac{100}{15} \rfloor = 6$ $N(P_3P_7) = \lfloor \frac{100}{21} \rfloor = 4$

$N(P_5P_7) = \lfloor \frac{100}{35} \rfloor = 2$

$N(P_2P_3P_5) = \lfloor \frac{100}{30} \rfloor = 3$ $N(P_2P_3P_7) = \lfloor \frac{100}{42} \rfloor = 2$

$N(P_2P_5P_7) = \lfloor \frac{100}{70} \rfloor = 1$ $N(P_3P_5P_7) = \lfloor \frac{100}{105} \rfloor = 0$ $N(P_2P_3P_5P_7) = \lfloor \frac{100}{210} \rfloor = 0$

Example 1, Section 8.6: Compute the number of solutions to $x_1+x_2+x_3 = 11$, where $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$, and $0 \leq x_3 \leq 6$ where each x_i is an integer.

Check the solution on page 586 in the textbook