Let A and B be two events. If the probability that B occurs is not affected by the occurence or non-occurence of A, then we say that A and B are **independent**.

Example: Consider two flips of a coin. Let A denote the event that the first flip is heads, and let B denote the event that the second flip is tails.

If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

.

$\Omega = \{TT, TH, HT, HH\}$

$A = \{HT, HH\}$ $B = \{TT, HT\}$

$A \cap B = \{HT\} \Rightarrow P(A \cap B) = \frac{1}{4}$

Since we know the events are independent, we can calculate $P(A \cap B)$ by multiplying $P(A)$ and $P(B)$.

$P(A) = .5$, $P(B) = .5$

$P(A) \cdot P(B) = .5 * .5 = .25$

Consider the random experiment where we flip a coin until we get a head

$$\Omega = \{ \underset{\frac{1}{2}}{H}, \underset{\frac{1}{4}}{TH}, \underset{\frac{1}{8}}{TTH}, \underset{\frac{1}{16}}{TTTH}, \dots \}$$

Probability that we flip heads first on the k^{th} flip?

$$\underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \dots \frac{1}{2}}_{k \text{ flips}} = \left(\frac{1}{2}\right)^k$$

What if we use a weighted coin that lands T with prob. 2/3 and H with prob. 1/3?

Prob. of first H on the kth flip?

$$\underbrace{TTT \cdots T}_{k-1} H$$

$$\underbrace{\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdots \frac{2}{3}}_{k-1} \cdot \frac{1}{3} = \left(\frac{2}{3}\right)^{k-1} \frac{1}{3}$$

Consider rolling a single die. Let A be the event that we roll a number larger than 3, and let B be the event that we roll an odd number. What is $P(A)$, $P(B)$, and $P(A \cap B)$?

$A = \{4,5,6\}$ $\quad$ $B = \{1,3,5\}$.

$P(A) = \frac{|A|}{|\Omega|} = \frac{3}{6} = \frac{1}{2}$

$P(B) = \frac{|B|}{|\Omega|} = \frac{3}{6} = \frac{1}{2}$

$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4}$ right?

~~wrong!~~ A and B are not independent, so we cannot apply this equation. P(A and B) = 1/6

If we know A occurred, then the possible outcomes are $\{4,5,6\}$. So now what is the probability that we rolled an odd number?

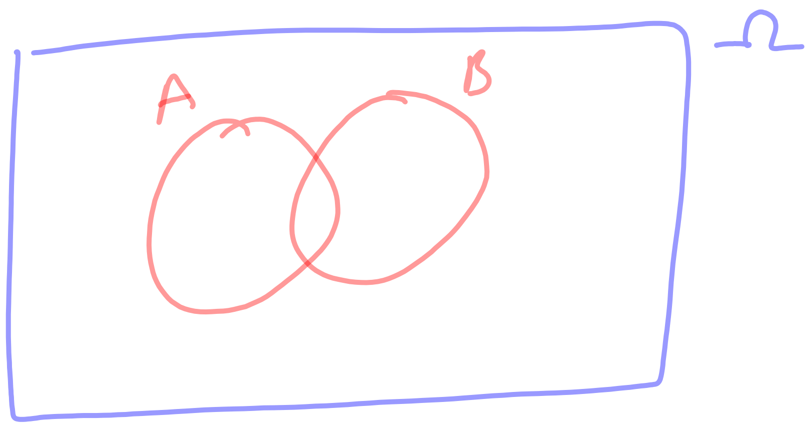
$$P(B|A) = \frac{|\{5\}|}{|\{4,5,6\}|} = \frac{1}{3}.$$

Without knowing A occurred, we have $P(B) = \frac{1}{2}$

Knowing that A occurred, we have $P(B) = \frac{1}{3}$.

Conditional Probability: Let $P(B|A)$ denote the probability that B occurs *given that* A occured.

Note that since we know that A has occured, the sample space is no longer all of Ω, and this changes the probability that B has also occured.



$$P(B|A) = \frac{|A \cap B|}{|A|} = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

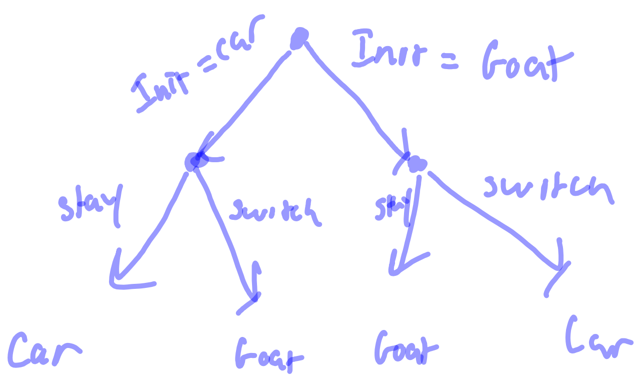
$$P(A \cap B) = P(B|A) \cdot P(A)$$

In our example $A = \{4,5,6\}$, and $B = \{1,3,5\}$.

$$P(A \cap B) = P(B|A) \cdot P(A) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Monte Hall problem: Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car; behind the others, goats. Once you choose your door, you are shown one of the doors that you did not choose that contains a goat. You are now given the option of staying on your door or switching to some other door. What should you do?

$$P(\text{Car} \mid \text{stay}) = ?$$



P(car I stay) = P(car I stay & initial choice is a car) * P(initial choice is a car) + P(car I stay & initial choice is a goat) * P(initial choice is a goat) = 1 * 1/3 + 0 * 2/3= 1/3

$$P(\text{Initial choice is car}) = \frac{1}{3},$$

$$P(\text{Initial choice is goat}) = \frac{2}{3}$$

P(car I switch) = P(car I switch & initial choice is a car) * P(initial choice is a car) + P(car I switch & initial choice is a goat) * P(initial choice is a goat) = 0 * 1/3 + 1 * 2/3= 2/3

Bayes' Theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Example: The players of a soccer league are tested for drugs using a special test. With this test, 98% of players who take steroids test positive. 12% of players not taking steroids test positive. It is estimated that 5% of all players take steroids. What is the probability that a player who tests positive takes steroids?

$P(A|B)$ where A = take steroids and
B = Players who test positive.

By Baye's Thm:

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)}$$

$P(B|A) = 0.98$, $P(A) = 0.05$

= P(A and B) + P((not A) and B) — Law of total probability
P(B) = P(A)*P(B|A) + P(not A)*P(B | not A)
= P(A)*P(B|A) + (1 - P(A))*P(B | not A)

$P(B) = 0.05 \times 0.98 + 0.95 \times .12 \approx 0.16$

$$P(A|B) = \frac{0.98 \times 0.05}{0.16} \approx 0.3$$