CS 3333: Mathematical Foundations

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- What is the order of $B \cdot A$?
  - It is 4×4.

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- **for** $i = 1 \to m$ **do**
    for $j = 1 \to n$ **do**
        $c_{ij} = 0$
        for $k = 1 \to p$ **do**
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
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- Matrix multiplication is associative.
  - $(A \cdot B) \cdot C = A \cdot (B \cdot C) = D$
- The order in which we do the multiplications can greatly impact the running time in computing D.
- When calculating D, is it better to compute $A \cdot B$ or $B \cdot C$ first?

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- We define A^0 to be I_n (the $n \times n$ identity matrix).

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- Example:
  - $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
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- In general, $A = (a_{ij})_{m \times n}$, $A^t = (a_{ji})_{n \times m}$.

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  - Example:
  - $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is a symmetric matrix.
- If L is a lower triangular matrix of size $n \times n$, then L^t is an upper triangular matrix of size $n \times n$ and vice versa.

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  - $|A| = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

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  - Write the first two columns again after the third column:

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

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- Consider the "downward diagonals":

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

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$$\begin{array}{ccc|cc} \color{red}{a_{11}} & \color{blue}{a_{12}} & \color{green}{a_{13}} & a_{11} & a_{12} \\ a_{21} & \color{red}{a_{22}} & \color{blue}{a_{23}} & \color{green}{a_{21}} & a_{22} \\ a_{31} & a_{32} & \color{red}{a_{33}} & \color{blue}{a_{31}} & \color{green}{a_{32}} \end{array}$$

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- The determinant is the sum of the products of the downward diagonals minus the sum of the products of the upward diagonals. This is called **Sarrus' rule** and only applies to 3×3 matrices.

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- $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

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  - $M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

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- The **cofactor** of a_{ij} is denoted $C_{ij} = (-1)^{i+j} M_{ij}$.
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- Pick any row or any column. Suppose we pick row 1. The determinant of A can be computed as $a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} \cdot C_{1n}$.
- This works for any row or any column (no matter which row or column we choose, we will get the same value).

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- $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$
  $$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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- Also, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$
  $-a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Exercise

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- Take the second row
- $|A| = -1 * \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 4 * \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 3 * \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$
- $|A| = -10$

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- $|A| = -10$
- Can you calculate it by picking another row or column?

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- Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
    - We can show that for any diagonal matrix D, the $|D|$ is simply the product of the elements along the diagonal.

More Matrices

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- Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
  - We can show that for any diagonal matrix D, the $|D|$ is simply the product of the elements along the diagonal.
  - Then $|A| = 0$, so the rank of A must be less than 3. Is there a 2×2 submatrix of A whose determinant is nonzero?

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  - $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a 2×2 submatrix of A whose determinant is 1.

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  - $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a 2×2 submatrix of A whose determinant is 1.
  - Therefore the rank of A is 2.

Example

- What is the **rank** of the following matrix A?
- $A = \begin{pmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{pmatrix}$

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  $\xrightarrow{r_4 = r_4 + r_3; r_2 = r_2 - 3r_1} \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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$$\xrightarrow{r_4=r_4+r_3;r_2=r_2-3r_1} \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_3=r_3+r_2} \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
- So, the rank is 2.