A **permutation** of a set of distinct elements is an ordered sequence/arrangement of these elements.

Example: Three students A, B, C stand in line for picture taking.

ABC
ACB
BAC
BCA
CAB
CBA

6 different permutations

$\underline{3} \cdot \underline{2} \cdot \underline{1} = 6$

An ordered arrangement of r elements out of n distinct elements is called an r**-permutation**. For example, in the previous problem we counted 3-permuations of 3 distinct elements..

$0 \leq r \leq n$ and $n \geq 0$

$P(n, r)$: number of r-permutations from n distinct elements.

What is $P(5,3)$? (Think of it as lining up 3 students from a group of 5.)

$\underline{5} \cdot \underline{4} \cdot \underline{3}$

Theorem 1: $P(n, r) = n(n-1)\cdots(n-r+1)$, for $1 \geq r \geq n, n > 0$.

$$\underset{1}{\underline{n}} \cdot \underset{2}{\underline{(n-1)}} \cdot \underset{3}{\underline{(n-2)}} \cdot \cdots\cdots \underset{r}{\underline{(n-r+1)}}$$

Corollary: $P(n, r) = n!/(n-r)!$.

$$\frac{n!}{(n-r)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\;\cdots\;(n-r+1)\cancel{(n-r)}\cancel{\cdot}\cdots\cdot 1}{\cancel{(n-r)}\cdot\cancel{(n-r-1)}\cdot(n-r-2)\cdot\;\cdots\;1}$$

Example: How many ways are there to select three winners (first place, second place, and third place) out of 100 people?

$P(n,r)$ where $n=100$ and $r=3$

$$P(100,3) = \underbrace{100\cdot 99\cdot 98}_{\text{Thm 1}} = \underbrace{\frac{100!}{97!}}_{\text{Corollary}}$$

Problem 2: $S = \{a, b, c, d, e, f, g\}. |S| = 7$. How many permuations of S are there?

$P(n,r)$ where $n=7$ and $r=7$

$P(7,7) = 7!$ $\qquad \frac{7!}{0!} = 7!$

Problem 2': How many 3-permuations of S are there?

$P(7,3) = 7 \cdot 6 \cdot 5 = 210$

$\frac{7!}{4!}$

Example 7: How many ways can the letters A, B, C, D, E, F, G, H be permuted to form 8-letter strings that contain the substring ABC?

Let $X = ABC$. We can reach the answer by finding all permutations of

D, E, F, G, H, X

$P(6,6) = 6!$

Combinations: How many different committees of three students can be formed from a group of 4? (Note that the order of their selection does not matter here.)

A, B, C, D

Committees: ABC
BCD
ABD
ACD

# of ways of choosing 3 students = # of ways of removing 1 student

If the order mattered, then this is $P(4,3) = 24$

An r-**combination** of the elements of a set is an unordered selection of r elements from the set.

Notation: $C(n,r)$ or $\binom{n}{r}$

"n choose r"

Theorem 2: $\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$

We can apply the division rule.

$P(n,r) = \frac{n!}{(n-r)!}$. This gives us many r-permutations, some of which we want to count as the same.

How many permutations are equivalent (should be counted as the same)? $r!$

$$\Rightarrow \binom{n}{r} = \frac{\left(\frac{n!}{(n-r)!}\right)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Example 11: The number of 5-card poker hands from a standard deck of 52 cards is:

$$\binom{52}{5} = \frac{52!}{(52-5)! \cdot 5!} = \frac{52!}{47!\, 5!}$$

What is the number of 47-card poker hands from a standard deck of 52 cards?

$$\binom{52}{47} = \frac{52!}{(52-47)!\, 47!} = \frac{52!}{5!\, 47!}$$

Corollary 2:

$$\binom{n}{r} = \binom{n}{n-r}, \quad r, n \text{ nonnegative}$$

Problem 33: A department has 10 men and 15 women. How many ways are there to form a committee of six if

1. There must be an equal number of men and women.

Must select 3 men and 3 women.

Ways of selecting men: $\binom{10}{3}$

Ways of selecting women: $\binom{15}{3}$

Product Rule: $\binom{10}{3}\binom{15}{3}$

2. There must be more women than men.

Choose 4 women, then any 2 from the rest.

$\binom{15}{4}\binom{21}{2}$

This will overcount some committees. Women A & B chosen in first round & C & D chosen in 2nd round counted differently than C & D in 1st and A & B in 2nd.

Committees could be: 4, 2 ; 5, 1; 6, 0

4,2: $\binom{15}{4}\binom{10}{2}$

5,1: $\binom{15}{5}\binom{10}{1}$

6,0: $\binom{15}{6}\binom{10}{0}$

Sum Rule: $\binom{15}{4}\binom{10}{2}+\binom{15}{5}\binom{10}{1}+\binom{15}{6}\binom{10}{0}$

Problem 35: Count the number of strings that contain exactly eight 0s and ten 1s if every 0 must be followed by a 1.

Since every 0 must be followed by a 1, we can view the problem as having the two tokens "01" and "1".

Since we have 8 0's, we must have 8 of these tokens

8 1's will come from the other token, so we need 2 of these.

10 positions: __ __ __ __ __ __ __ __ __ __

We must choose 2 positions for the "1" tokens. There are $\binom{10}{2}$ ways of doing this. Equivalently we could choose the 8 positions for the "01" tokens. $\binom{10}{8}$

$$\binom{10}{2} = \frac{10!}{8!2!} = \frac{10 \cdot 9}{2!} = 5 \cdot 9 = 45$$