CS 3333: Mathematical Foundations

Binomial Theorem

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Consider all 4-bit strings

0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
1	1	0	0
1	1	0	1
1	1	1	0
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Product rule:
$$2 * 2 * 2 * 2 = 2^4 = 16$$

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Product rule: $2 * 2 * 2 * 2 = 2^4 = 16$ Define $n_i = \#$ of bit strings with *i* 1s. $n_0 = \binom{4}{0} = 1;$ $n_1 = \binom{4}{1} = 4;$ $n_2 = \binom{4}{2} = 6;$ $n_3 = \binom{4}{3} = 4;$ $n_4 = \binom{4}{4} = 1;$

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	$n_0 = \binom{4}{0} = 1;$			
	$n_1 = \binom{4}{1} = 4;$			
	$n_2 = \binom{4}{2} = 6;$			
	$n_3 = \binom{4}{3} = 4;$			
	$n_4={4 \choose 4}=1;$			
The	total is			
	$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$			
	= 1 + 4 + 6 + 4 + 1 = 16			

- Binomial expression: a sum of two terms, e.g., x + y.
- Powers of binomial expressions: $(x + y)^4$, $(x + y)^3$, $(a b)^{10}$.

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Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

, where
$$n \ge 0$$
.

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}y^n$$

$$= \sum_{k=0}^n \binom{n}{k}x^{n-k}y^k$$

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$$= \binom{3}{0}x^{3} + \binom{3}{1}x^{2}y + \binom{3}{2}xy^{2} + \binom{3}{3}y^{3}$$

• What is the coefficient of $x^{12} \cdot y^{13}$ in $(x + y)^{25}$?

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- 13 times of y. $\binom{25}{13}x^{12} \cdot y^{13}$.
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- Assume X = 2x and Y = -3y, then $(X + Y)^{25}$.

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- The term is
- $\blacktriangleright = \binom{25}{12} 2^{12} (-3)^{13} x^{12} \cdot y^{13}$
- The coefficient is $-\binom{25}{12}2^{12}\cdot 3^{13}$

- Corollary 1: $\sum_{k=0}^{n} {n \choose k} = 2^{n}$
- Apply the binomial theorem on $(x + y)^n$ when x = y = 1.

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$$(x+y)^n = (1+1)^n = 2^n$$

Corollary 1: $\sum_{k=0}^{n} {n \choose k} = 2^{n}$ Apply the binomial theorem on $(x + y)^{n}$ when x = y = 1. $(x + y)^{n} = (1 + 1)^{n} = 2^{n}$ $(1 + 1)^{n}$ $= {n \choose 0} 1^{n} \cdot 1^{0} + {n \choose 1} 1^{n-1} \cdot 1^{1} + {n \choose 2} 1^{n-2} \cdot 1^{2} + \dots + {n \choose n} 1^{0} \cdot 1^{n}$ $= {n \choose 0} + {n \choose 1} + {n \choose 2} + \dots + {n \choose n}$ Therefore, $\sum_{k=0}^{n} {n \choose k} = 2^{n}$.

• Corollary 2:
$$\sum_{k=0}^{n} (-1)^k {n \choose k} = 0$$

Apply the binomial theorem on $(x + y)^n$ when x = 1, y = -1.

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$$(x+y)^n = (1-1)^n = 0$$

Corollary 2:
$$\sum_{k=0}^{n} (-1)^{k} {n \choose k} = 0$$
Apply the binomial theorem on $(x + y)^{n}$ when $x = 1, y = -1$.
 $(x + y)^{n} = (1 - 1)^{n} = 0$
 $(1 - 1)^{n}$
 $= {n \choose 0} 1^{n} \cdot (-1)^{0} + {n \choose 1} 1^{n-1} \cdot (-1)^{1} + \ldots + {n \choose n} 1^{0} \cdot (-1)^{n}$
 $= {n \choose 0} - {n \choose 1} + {n \choose 2} - \ldots + (-1)^{n} {n \choose n} = 0$
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Therefore, $\sum_{k=0}^{n} (-1)^{k} {n \choose k} = 0$.
 $n = 5, (1 - 1)^{5} = {5 \choose 0} - {5 \choose 1} + {5 \choose 2} - {5 \choose 3} + {5 \choose 4} - {5 \choose 5} = 1 - 5 + 10 - 10 + 5 - 1 = 0$
 $n = 6, (1 - 1)^{6} = {6 \choose 0} - {6 \choose 1} + {6 \choose 2} - {6 \choose 3} + {6 \choose 4} - {6 \choose 5} + {6 \choose 6} = 1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$

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Pascal's Identity

$$\blacktriangleright \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

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Pascal's Identity

 $\triangleright \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ Pascal's triangle $\begin{pmatrix} 0\\ 0 \end{pmatrix}$ 1 $\binom{1}{0}\binom{1}{1}$ 1 1 $\binom{2}{0}\binom{2}{1}\binom{2}{2}$ 2 1 By Pascal's identity: 1 $\begin{pmatrix} 3\\0 \end{pmatrix} \begin{pmatrix} 3\\1 \end{pmatrix} \begin{pmatrix} 3\\2 \end{pmatrix} \begin{pmatrix} 3\\3 \end{pmatrix}$ $\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$ 1 3 3 1 $\begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ 4 6 4 1 $\begin{pmatrix} 5\\0 \end{pmatrix} \begin{pmatrix} 5\\1 \end{pmatrix} \begin{pmatrix} 5\\2 \end{pmatrix} \begin{pmatrix} 5\\3 \end{pmatrix} \begin{pmatrix} 5\\4 \end{pmatrix} \begin{pmatrix} 5\\5 \end{pmatrix}$ 5 10 10 5 $\begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix}$ 15 20 15 $\begin{pmatrix} 7\\0 \end{pmatrix} \begin{pmatrix} 7\\1 \end{pmatrix} \begin{pmatrix} 7\\2 \end{pmatrix} \begin{pmatrix} 7\\3 \end{pmatrix} \begin{pmatrix} 7\\3 \end{pmatrix} \begin{pmatrix} 7\\4 \end{pmatrix} \begin{pmatrix} 7\\5 \end{pmatrix} \begin{pmatrix} 7\\6 \end{pmatrix} \begin{pmatrix} 7\\7 \end{pmatrix}$ 21 35 35 21 7 7 $\binom{8}{0}$ $\binom{8}{1}$ $\binom{8}{2}$ $\binom{8}{3}$ $\binom{8}{4}$ $\binom{8}{5}$ $\binom{8}{6}$ $\binom{8}{7}$ $\binom{8}{8}$ 1 28 56 70 56 28

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Proof of Pascal's Identity

$$\binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!}$$

$$= \frac{n!\cdot k}{(n-k+1)!(k-1)!\cdot k} + \frac{n!\cdot(n-k+1)}{(n-k)!\cdot(n-k+1)k!}$$

$$= \frac{n!(k+n-k+1)}{(n-k+1)!k!}$$

$$= \frac{(n+1)!}{(n+1-k)!k!}$$

$$= \binom{n+1}{k}$$

Password Counting Problem

Revisit the password counting problem. If a length 6 password can consist of a lowercase letters and digits, how many passwords are there that contain at least one digit?

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► 36⁶ - 26⁶



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- Revisit the password counting problem. If a length 6 password can consist of a lowercase letters and digits, how many passwords are there that contain at least one digit?
- ▶ 36⁶ 26⁶
- If we consider the number of digits:
- 0 digit: $\binom{6}{0} 10^0 \cdot 26^6$
- ▶ 1 digits: $\binom{6}{1}10^1 \cdot 26^5$
- ▶ 2 digits: $\binom{6}{2} 10^2 \cdot 26^4$
- 3 digits: $\binom{6}{3} 10^3 \cdot 26^3$
- 4 digits: $\binom{6}{4} 10^4 \cdot 26^2$
- 5 digits: $\binom{6}{5}10^5 \cdot 26^1$
- 6 digits: $\binom{6}{6} 10^6 \cdot 26^0$
- The total is $(10 + 26)^6$ by the binomial theorem.

• Then, we get $36^6 - 26^6$.

Theorem 3: Vandermonde's Identity (VI) [K.R. 6.4.3]

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

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Combinatorial proof:

▶ $0 \le r \le min(m, n)$.

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For example, we select a committee of r people from m men and n women.

• The direct answer is to select r people from m + n people, $\binom{m+n}{r}$

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Theorem 3: Vandermonde's Identity (VI) [K.R. 6.4.3]

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

▶ Wikipedia

Combinatorial proof:

▶ $0 \le r \le min(m, n)$.

- For example, we select a committee of r people from m men and n women.
- The direct answer is to select *r* people from m + n people, $\binom{m+n}{r}$
- ▶ Or, we select $k (\leq m)$ people from *m* men, and select r k people from *n* women, where $k \in [0, r]$.

Theorem 3: Vandermonde's Identity (VI)

•
$$0 \le r \le \min(m, n)$$
.
• For example, $m = 5$, $n = 7$, and $r = 4$.
• $\binom{5+7}{4} = 495$
• $\sum_{k=0}^{4} \binom{5}{k} \binom{7}{4-k} = \binom{5}{0} \binom{7}{4-0} + \binom{5}{1} \binom{7}{4-1} + \binom{5}{2} \binom{7}{4-2} + \binom{5}{3} \binom{7}{4-3} + \binom{5}{4} \binom{7}{4-4}$
• $= 35 + 5 * 35 + 10 * 21 + 10 * 7 + 5$
• $= 495$

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• Method 2: First, choose the chairman from *n* people. Then choose the remains k - 1 from n - 1 people.

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Formal proof:

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•
$$LHS = k \binom{n}{k} = k \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!}$$

•
$$RHS = n\binom{n-1}{k-1} = n\frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$$

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Proved it.