Section 2.4
Writing Direct Proofs

Direct Proofs

- To prove an implication $P(x) \rightarrow Q(x)$, we assume $P(x)$ and from it derive $Q(x)$
- Note that implications often occur as part of a universal statement
  - "If x is an odd integer then x^2 is an odd integer" by itself has a hidden quantifier
  - "For all x, if x is an odd integer then x^2 is an odd integer"
  - We must prove this statement for an arbitrary x for which we make no assumptions other than those in the statement (x is an odd integer)

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k

    7. x^2 is an odd integer

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k

    6. $x^2 = 2j + 1$ for some integer j
    7. x^2 is an odd integer

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k
    3. $x^2 = (2k + 1)(2k + 1)$

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k
    3. $x^2 = (2k + 1)(2k + 1)$
    4. $x^2 = 4k^2 + 4k + 1$

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k
    3. $x^2 = (2k + 1)(2k + 1)$
    4. $x^2 = 4k^2 + 4k + 1$
    5. $x^2 = 2\left(2k^2 + 2k\right) + 1$ where $2k^2 + 2k$ is an integer

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k
    3. $x^2 = (2k + 1)(2k + 1)$
    4. $x^2 = 4k^2 + 4k + 1$
    5. $x^2 = 2(2k^2 + 2k) + 1$ where $2k^2 + 2k$ is an integer
    6. $x^2 = 2j + 1$ where $j = 2k^2 + 2k$ is an integer

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k
    3. $x^2 = (2k + 1)(2k + 1)$
    4. $x^2 = 4k^2 + 4k + 1$
    5. $x^2 = 2\left(2k^2 + 2k\right) + 1$ where $2k^2 + 2k$ is an integer
    6. $x^2 = 2j + 1$ where $j = 2k^2 + 2k$ is an integer
    7. x^2 is an odd integer

Direct Proofs

- Example: If x is an odd integer then x^2 is an odd integer
  - Proof:
    1. Assume x is an odd integer
    2. $x = 2k + 1$ for some integer k
    3. $x^2 = (2k + 1)(2k + 1)$
    4. $x^2 = 4k^2 + 4k + 1$
    5. $x^2 = 2\left(2k^2 + 2k\right) + 1$ where $2k^2 + 2k$ is an integer
    6. $x^2 = 2j + 1$ where $j = 2k^2 + 2k$ is an integer
    7. x^2 is an odd integer
    8. If x is an odd integer then x^2 is an odd integer

Direct Proofs

- Watch the video on direct proofs in section 2.4

Direct Proofs of Rational Numbers

- A number r is rational if there are integers x and y such that:
  - $y \neq 0$ and
  - $r = \frac{x}{y}$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$

  8. $r + s$ is a rational number

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $s = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$

  7. $r + s = \frac{x}{y}$ where x is an integer, y is an integer, and $y \neq 0$
  8. $r + s$ is a rational number

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
  3. $r + s = \frac{a}{b} + \frac{c}{d}$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
  3. $r + s = \frac{a}{b} + \frac{c}{d}$
  4. $r + s = \frac{d}{d}\left(\frac{a}{b}\right) + \frac{b}{b}\left(\frac{c}{d}\right)$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
  3. $r + s = \frac{a}{b} + \frac{c}{d}$
  4. $r + s = \frac{d}{d}\left(\frac{a}{b}\right) + \frac{b}{b}\left(\frac{c}{d}\right)$
  5. $r + s = \frac{ad}{bd} + \frac{bc}{bd}$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
  3. $r + s = \frac{a}{b} + \frac{c}{d}$
  4. $r + s = \frac{d}{d}\left(\frac{a}{b}\right) + \frac{b}{b}\left(\frac{c}{d}\right)$
  5. $r + s = \frac{ad}{bd} + \frac{bc}{bd}$
  6. $r + s = \frac{ad+bc}{bd}$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
  3. $r + s = \frac{a}{b} + \frac{c}{d}$
  4. $r + s = \frac{d}{d}\left(\frac{a}{b}\right) + \frac{b}{b}\left(\frac{c}{d}\right)$
  5. $r + s = \frac{ad}{bd} + \frac{bc}{bd}$
  6. $r + s = \frac{ad+bc}{bd}$
  7. $r + s = \frac{x}{y}$ where $x = ad + bc$ is an integer, $y = bd$ is an integer, and $bd \neq 0$

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

  Proof:

  1. Assume r and s are rational numbers
  2. $r = \frac{a}{b}$ and $s = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
  3. $r + s = \frac{a}{b} + \frac{c}{d}$
  4. $r + s = \frac{d}{d}\left(\frac{a}{b}\right) + \frac{b}{b}\left(\frac{c}{d}\right)$
  5. $r + s = \frac{ad}{bd} + \frac{bc}{bd}$
  6. $r + s = \frac{ad+bc}{bd}$
  7. $r + s = \frac{x}{y}$ where $x = ad + bc$ is an integer, $y = bd$ is an integer, and $bd \neq 0$
  8. $r + s$ is a rational number

Direct Proofs of Rational Numbers

- Example if r and s are rational numbers, then $r + s$ is a rational number

Proof:

1. Assume r and s are rational numbers
2. $r = \frac{a}{b}$ and $\mathrm{s} = \frac{c}{d}$ where a, b, c, and d are integers, $b \neq 0$, and $d \neq 0$
3. $r + s = \frac{a}{b} + \frac{c}{d}$
4. $r + s = \frac{d}{d}\left(\frac{a}{b}\right) + \frac{b}{b}\left(\frac{c}{d}\right)$
5. $r + s = \frac{ad}{bd} + \frac{bc}{bd}$
6. $r + s = \frac{ad+bc}{bd}$
7. $r + s = \frac{x}{y}$ where $x = ad + bc$ is an integer, $y = bd$ is an integer, and $bd \neq 0$
8. $r + s$ is a rational number
9. If r and s are rational numbers, then $r + s$ is a rational number

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i
  4. $c = bj$ for some integer j

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i
  4. $c = bj$ for some integer j
  5. $c = aij$ where ij is an integer

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i
  4. $c = bj$ for some integer j
  5. $c = aij$ where ij is an integer
  6. $c = ak$ where k is an integer

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i
  4. $c = bj$ for some integer j
  5. $c = aij$ where ij is an integer
  6. $c = ak$ where k is an integer
  7. a divides c

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i
  4. $c = bj$ for some integer j
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  6. $c = ak$ where k is an integer
  7. a divides c
  8. If a divides b and b divides c then a divides c

Direct Proofs of Divisibility

- Example if a, b, and c are integers, then if a divides b and b divides c, then a divides c

  Proof:

  1. Assume a, b, and c are integers
  2. Assume a divides b and b divides c
  3. $b = ai$ for some integer i
  4. $c = bj$ for some integer j
  5. $c = aij$ where ij is an integer
  6. $c = ak$ where k is an integer
  7. a divides c
  8. If a divides b and b divides c then a divides c
  9. if a, b, and c are integers, then if a divides b and b divides c, then a divides c

Section 2.5
Proofs by Contrapositive

Proofs by Contrapositive

- If it is difficult to prove $p \to q$ by a direct proof, we can instead prove $p \to q$ by a proof by contrapositive
- Since $p \to q \equiv \neg q \to \neg p$, we can prove $p \to q$ by proving $\neg q \to \neg p$ by using a direct proof

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
    1. Assume that n is an integer and n is not odd

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
  1. Assume that n is an integer and n is not odd
  2. n is even

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
  1. Assume that n is an integer and n is not odd
  2. n is even
  3. $n = 2k$ for some integer k

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
  1. Assume that n is an integer and n is not odd
  2. n is even
  3. $n = 2k$ for some integer k
  4. $3n + 2 = 6k + 2$

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
  1. Assume that n is an integer and n is not odd
  2. n is even
  3. $n = 2k$ for some integer k
  4. $3n + 2 = 6k + 2$
  5. $3n + 2 = 2(3k + 1)$ where $(3k + 1)$ is an integer because k is an integer

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
    1. Assume that n is an integer and n is not odd
    2. n is even
    3. $n = 2k$ for some integer k
    4. $3n + 2 = 6k + 2$
    5. $3n + 2 = 2(3k + 1)$ where $(3k + 1)$ is an integer because k is an integer
    6. $3n + 2$ is even

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
    1. Assume that n is an integer and n is not odd
    2. n is even
    3. $n = 2k$ for some integer k
    4. $3n + 2 = 6k + 2$
    5. $3n + 2 = 2(3k + 1)$ where $(3k + 1)$ is an integer because k is an integer
    6. $3n + 2$ is even
    7. $3n + 2$ is not odd

Proofs by Contrapositive

- Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- This is difficult to prove using a direct proof, so instead prove that if n is not odd, then $3n + 2$ is not odd
  1. Assume that n is an integer and n is not odd
  2. n is even
  3. $n = 2k$ for some integer k
  4. $3n + 2 = 6k + 2$
  5. $3n + 2 = 2(3k + 1)$ where $(3k + 1)$ is an integer because k is an integer
  6. $3n + 2$ is even
  7. $3n + 2$ is not odd
  8. if n is an integer and $3n + 2$ is odd, then n is odd

Proofs by Contrapositive

- Another Example: Prove that if $n = ab$ where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
- Rephrase: If a and b are positive integers then if $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
- Replace inner implication with its contrapositive:

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
  3. By De Morgan's law: $a > \sqrt{n}$ and $b > \sqrt{n}$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
  3. By De Morgan's law: $a > \sqrt{n}$ and $b > \sqrt{n}$
  4. $ab > \sqrt{n}\sqrt{n}$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
  3. By De Morgan's law: $a > \sqrt{n}$ and $b > \sqrt{n}$
  4. $ab > \sqrt{n}\sqrt{n}$
  5. $ab > n$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
  3. By De Morgan's law: $a > \sqrt{n}$ and $b > \sqrt{n}$
  4. $ab > \sqrt{n}\sqrt{n}$
  5. $ab > n$
  6. It is not the case that $n = ab$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
  3. By De Morgan's law: $a > \sqrt{n}$ and $b > \sqrt{n}$
  4. $ab > \sqrt{n}\sqrt{n}$
  5. $ab > n$
  6. It is not the case that $n = ab$
  7. If it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

Proofs by Contrapositive

- Proof of

  If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$

  1. Assume a and b are positive integers
  2. Assume it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
  3. By De Morgan's law: $a > \sqrt{n}$ and $b > \sqrt{n}$
  4. $ab > \sqrt{n}\sqrt{n}$
  5. $ab > n$
  6. It is not the case that $n = ab$
  7. If it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$
  8. If a and b are positive integers, then if it is not the case that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ then it is not the case that $n = ab$