Section 2.7
Proof By Cases

Proofs by Cases

- Proofs by cases are generalizations of exhaustive proofs
- Instead of considering specific values, specific categories of values are used
- The categories must be exhaustive; i.e. they must cover all situations applicable to the theorem being proved

Proofs by Cases

- Example: Prove that if n is an integer, then $n^2 \geq n$.
- Proof by cases
    1. Assume n is an integer

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Proofs by Cases

- Example: Prove that if n is an integer, then $n^2 \geq n$.
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  1. Assume n is an integer
  2. Since n is an integer, it falls into one of 3 categories: $n < 0, n = 0, n > 0$
  3. Case1: Assume that $n < 0$

Proofs by Cases

- Example: Prove that if n is an integer, then $n^2 \geq n$.
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  1. Assume n is an integer
  2. Since n is an integer, it falls into one of 3 categories: $n < 0, n = 0, n > 0$
  3. Case1: Assume that $n < 0$
  4. n^2 is a positive integer

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- Example: Prove that if n is an integer, then $n^2 \geq n$.
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  1. Assume n is an integer
  2. Since n is an integer, it falls into one of 3 categories: $n < 0, n = 0, n > 0$
  3. Case1: Assume that $n < 0$
  4. n^2 is a positive integer
  5. $n < 0 \leq n^2$

Proofs by Cases

- Example: Prove that if n is an integer, then $n^2 \geq n$.
- Proof by cases

1. Assume n is an integer
2. Since n is an integer, it falls into one of 3 categories: $n < 0, n = 0, n > 0$
3. Case1: Assume that $n < 0$
4. n^2 is a positive integer
5. $n < 0 \leq n^2$
6. $n^2 \geq n$

Proofs by Cases

- Example: Prove that if n is an integer, then $n^2 \geq n$.
- Proof by cases
  1. Assume n is an integer
  2. Since n is an integer, it falls into one of 3 categories: $n < 0$, $n = 0$, $n > 0$
  3. Case1: Assume that $n < 0$
  4. n^2 is a positive integer
  5. $n < 0 \leq n^2$
  6. $n^2 \geq n$
  7. Therefore, if $n < 0$, then $n^2 \geq n$

Proofs by Cases

- Proof by cases continued
  8. Case2: Assume that $n = 0$

Proofs by Cases

- Proof by cases continued
    8. Case2: Assume that $n = 0$
    9. $n^2 = 0 \geq 0$

Proofs by Cases

- Proof by cases continued

8. Case2: Assume that $n = 0$
9. $n^2 = 0 \geq 0$
10. $n^2 \geq n$

Proofs by Cases

- Proof by cases continued

  8. Case2: Assume that $n = 0$
  9. $n^2 = 0 \geq 0$
  10. $n^2 \geq n$
  11. Therefore, if $n = 0$, then $n^2 \geq n$

Proofs by Cases

- Proof by cases continued

8. Case2: Assume that $n = 0$
9. $n^2 = 0 \geq 0$
10. $n^2 \geq n$
11. Therefore, if $n = 0$, then $n^2 \geq n$
12. Case3: Assume that $n > 0$

Proofs by Cases

- Proof by cases continued
  8. Case2: Assume that $n = 0$
  9. $n^2 = 0 \geq 0$
  10. $n^2 \geq n$
  11. Therefore, if $n = 0$, then $n^2 \geq n$
  12. Case3: Assume that $n > 0$
  13. $n \geq 1$

Proofs by Cases

- Proof by cases continued

8. Case2: Assume that $n = 0$
9. $n^2 = 0 \geq 0$
10. $n^2 \geq n$
11. Therefore, if $n = 0$, then $n^2 \geq n$
12. Case3: Assume that $n > 0$
13. $n \geq 1$
14. $n \cdot n \geq n \cdot 1$

Proofs by Cases

- Proof by cases continued

8. Case2: Assume that $n = 0$
9. $n^2 = 0 \geq 0$
10. $n^2 \geq n$
11. Therefore, if $n = 0$, then $n^2 \geq n$
12. Case3: Assume that $n > 0$
13. $n \geq 1$
14. $n \cdot n \geq n \cdot 1$
15. $n^2 \geq n$

Proofs by Cases

- Proof by cases continued

8. Case2: Assume that $n = 0$
9. $n^2 = 0 \geq 0$
10. $n^2 \geq n$
11. Therefore, if $n = 0$, then $n^2 \geq n$
12. Case3: Assume that $n > 0$
13. $n \geq 1$
14. $n \cdot n \geq n \cdot 1$
15. $n^2 \geq n$
16. Therefore, if $n > 0$, then $n^2 \geq n$

Proofs by Cases

- Proof by cases continued

8. Case2: Assume that $n = 0$
9. $n^2 = 0 \geq 0$
10. $n^2 \geq n$
11. Therefore, if $n = 0$, then $n^2 \geq n$
12. Case3: Assume that $n > 0$
13. $n \geq 1$
14. $n \cdot n \geq n \cdot 1$
15. $n^2 \geq n$
16. Therefore, if $n > 0$, then $n^2 \geq n$
17. Therefore, if n is an integer, then $n^2 \geq n$

Proofs by Cases

- Another example: Prove that $|x \cdot y| = |x| \cdot |y|$
- Proof by cases
    1. Assume that x and y are real numbers

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  1. Assume that x and y are real numbers
  2. There are 4 possible cases for x and y: 1) x and y are both non-negative, 2) x is non-negative and y is negative, 3) x is negative and y is non-negative, 4) x and y are both negative

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  3. Case 1: Assume that x and y are both non-negative

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  3. Case 1: Assume that x and y are both non-negative
  4. $\quad |x \cdot y| = x \cdot y = |x| \cdot |y|$

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    3. Case 1: Assume that x and y are both non-negative
    4. $|x \cdot y| = x \cdot y = |x| \cdot |y|$
    5. Therefore, if x and y are both non-negative, then $|x \cdot y| = |x| \cdot |y|$

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.
    6. Case 2: Assume that x is non-negative and y is negative

Proofs by Cases

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6. Case 2: Assume that x is non-negative and y is negative
7. $|x \cdot y| = -(x \cdot y) = x \cdot -y = x \cdot |y| = |x| \cdot |y|$

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

6. Case 2: Assume that x is non-negative and y is negative
7. $|x \cdot y| = -(x \cdot y) = x \cdot -y = x \cdot |y| = |x| \cdot |y|$
8. Therefore, if x is non-negative and y is negative, then $|x \cdot y| = |x| \cdot |y|$

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

6. Case 2: Assume that x is non-negative and y is negative
7. $|x \cdot y| = -(x \cdot y) = x \cdot -y = x \cdot |y| = |x| \cdot |y|$
8. Therefore, if x is non-negative and y is negative, then $|x \cdot y| = |x| \cdot |y|$
9. Case 3: Assume that x is negative and y is non-negative

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

6. Case 2: Assume that x is non-negative and y is negative
7. $\quad |x \cdot y| = -(x \cdot y) = x \cdot -y = x \cdot |y| = |x| \cdot |y|$
8. $\quad$ Therefore, if x is non-negative and y is negative, then $|x \cdot y| = |x| \cdot |y|$
9. Case 3: Assume that x is negative and y is non-negative
10. $\quad |x \cdot y| = -(x \cdot y) = -x \cdot y = |x| \cdot y = |x| \cdot |y|$

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- Except for assumptions, each line below follows from the lines above it.

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7. $\quad |x \cdot y| = -(x \cdot y) = x \cdot -y = x \cdot |y| = |x| \cdot |y|$
8. $\quad$ Therefore, if x is non-negative and y is negative, then $|x \cdot y| = |x| \cdot |y|$
9. Case 3: Assume that x is negative and y is non-negative
10. $\quad |x \cdot y| = -(x \cdot y) = -x \cdot y = |x| \cdot y = |x| \cdot |y|$
11. $\quad$ Therefore, if x is negative and y is non-negative, then $|x \cdot y| = |x| \cdot |y|$

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

    12. Case 4: Assume that x and y are both negative

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

  12. Case 4: Assume that x and y are both negative
  13. $|x \cdot y| = x \cdot y = -x \cdot -y = |x| \cdot |y|$

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

  12. Case 4: Assume that x and y are both negative

  13. $|x \cdot y| = x \cdot y = -x \cdot -y = |x| \cdot |y|$

  14. Therefore, if x and y are both negative, then $|x \cdot y| = |x| \cdot |y|$

Proofs by Cases

- Except for assumptions, each line below follows from the lines above it.

12. Case 4: Assume that x and y are both negative
13. $|x \cdot y| = x \cdot y = -x \cdot -y = |x| \cdot |y|$
14. Therefore, if x and y are both negative, then $|x \cdot y| = |x| \cdot |y|$
15. Therefore, $|x \cdot y| = |x| \cdot |y|$

Without Loss of Generality

- In the previous proof there were two cases that were extremely similar:
  - x is non-negative and y is negative
  - x is negative and y is non-negative
  - In both cases there is a non-negative integer and a negative integer
  - They are essentially the same because $x \cdot y = y \cdot x$
- These two cases could have been combined together without a loss of generality by considering just one of the two

Without Loss of Generality

- Example: For any integers x and y, if x is even or y is even, then xy is even.
- Proof
    1. Assume that x and y are integers and x is even or y is even

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- Proof
  1. Assume that x and y are integers and x is even or y is even
  2. Without loss of generality, assume that x is even

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  1. Assume that x and y are integers and x is even or y is even
  2. Without loss of generality, assume that x is even
  3. $x = 2j$ for some integer j

Without Loss of Generality

- Example: For any integers x and y, if x is even or y is even, then xy is even.
- Proof
    1. Assume that x and y are integers and x is even or y is even
    2. Without loss of generality, assume that x is even
    3. $x = 2j$ for some integer j
    4. $xy = 2jy$ where jy is an integer

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- Example: For any integers x and y, if x is even or y is even, then xy is even.
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  1. Assume that x and y are integers and x is even or y is even
  2. Without loss of generality, assume that x is even
  3. $x = 2j$ for some integer j
  4. $xy = 2jy$ where jy is an integer
  5. xy is even

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- Proof
  1. Assume that x and y are integers and x is even or y is even
  2. Without loss of generality, assume that x is even
  3. $x = 2j$ for some integer j
  4. $xy = 2jy$ where jy is an integer
  5. xy is even
  6. if x is even or y is even, then xy is even.

Without Loss of Generality

- Another example: Prove that if x and y are integers, and both xy and $x + y$ are even, then both x and y are even.
- Proof
  1. For a proof by contraposition, assume it is not the case that both x and y are even

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  1. For a proof by contraposition, assume it is not the case that both x and y are even
  2. At least one of x and y is odd. Without loss of generality, assume that x is odd.

Without Loss of Generality

- Another example: Prove that if x and y are integers, and both xy and $x + y$ are even, then both x and y are even.
- Proof
  1. For a proof by contraposition, assume it is not the case that both x and y are even
  2. At least one of x and y is odd. Without loss of generality, assume that x is odd.
  3. There are two cases for y: y is even and y is odd

Without Loss of Generality

4. Case 1: Assume y is even

Without Loss of Generality

4. Case 1: Assume y is even
5. $x = 2i + 1$ for some integer i

Without Loss of Generality

4. Case 1: Assume y is even
5. $\quad x = 2i + 1$ for some integer i
6. $\quad y = 2k$ for some integer k

Without Loss of Generality

4. Case 1: Assume y is even
5. $\quad x = 2i + 1$ for some integer i
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7. $\quad x + y = 2i + 1 + 2k$

Without Loss of Generality

4. Case 1: Assume y is even
5. $x = 2i + 1$ for some integer i
6. $y = 2k$ for some integer k
7. $x + y = 2i + 1 + 2k$
8. $x + y = 2i + 2k + 1$

Without Loss of Generality

4. Case 1: Assume y is even
5. $\quad x = 2i + 1$ for some integer i
6. $\quad y = 2k$ for some integer k
7. $\quad x + y = 2i + 1 + 2k$
8. $\quad x + y = 2i + 2k + 1$
9. $\quad x + y = 2(i + k) + 1$ where $i + k$ is an integer

Without Loss of Generality

4. Case 1: Assume y is even
5. $x = 2i + 1$ for some integer i
6. $y = 2k$ for some integer k
7. $x + y = 2i + 1 + 2k$
8. $x + y = 2i + 2k + 1$
9. $x + y = 2(i + k) + 1$ where $i + k$ is an integer
10. $x + y$ is odd

Without Loss of Generality

4. Case 1: Assume y is even
5. $x = 2i + 1$ for some integer i
6. $y = 2k$ for some integer k
7. $x + y = 2i + 1 + 2k$
8. $x + y = 2i + 2k + 1$
9. $x + y = 2(i + k) + 1$ where $i + k$ is an integer
10. $x + y$ is odd
11. It is not the case that both xy and $x + y$ are even

Without Loss of Generality

4. Case 1: Assume y is even
5. $x = 2i + 1$ for some integer i
6. $y = 2k$ for some integer k
7. $x + y = 2i + 1 + 2k$
8. $x + y = 2i + 2k + 1$
9. $x + y = 2(i + k) + 1$ where $i + k$ is an integer
10. $x + y$ is odd
11. It is not the case that both xy and $x + y$ are even
12. Therefore, if y is even, then it is not the case that both xy and $x + y$ are even

Without Loss of Generality

13. Case 2: Assume y is odd

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Without Loss of Generality

13. Case 2: Assume y is odd
14. $\quad x = 2i + 1$ for some integer i

Without Loss of Generality

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14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k

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Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k
16. $xy = (2i + 1)(2k + 1)$

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Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k
16. $xy = (2i + 1)(2k + 1)$
17. $xy = 4ik + 2i + 2k + 1$

.

Without Loss of Generality

13. Case 2: Assume y is odd
14. $\quad x = 2i + 1$ for some integer i
15. $\quad y = 2k + 1$ for some integer k
16. $\quad xy = (2i + 1)(2k + 1)$
17. $\quad xy = 4ik + 2i + 2k + 1$
18. $\quad xy = 2(2ik + i + k) + 1$

.

Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k
16. $xy = (2i + 1)(2k + 1)$
17. $xy = 4ik + 2i + 2k + 1$
18. $xy = 2(2ik + i + k) + 1$
19. xy is odd

.

Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k
16. $xy = (2i + 1)(2k + 1)$
17. $xy = 4ik + 2i + 2k + 1$
18. $xy = 2(2ik + i + k) + 1$
19. xy is odd
20. It is not the case that both xy and $x + y$ are even

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Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k
16. $xy = (2i + 1)(2k + 1)$
17. $xy = 4ik + 2i + 2k + 1$
18. $xy = 2(2ik + i + k) + 1$
19. xy is odd
20. It is not the case that both xy and $x + y$ are even
21. Therefore, if y is odd, then it is not the case that both xy and $x + y$ are even

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Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
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17. $xy = 4ik + 2i + 2k + 1$
18. $xy = 2(2ik + i + k) + 1$
19. xy is odd
20. It is not the case that both xy and $x + y$ are even
21. Therefore, if y is odd, then it is not the case that both xy and $x + y$ are even
22. If it is not the case that both x and y are even, then it is not the case that both xy and $x + y$ are even

Without Loss of Generality

13. Case 2: Assume y is odd
14. $x = 2i + 1$ for some integer i
15. $y = 2k + 1$ for some integer k
16. $xy = (2i + 1)(2k + 1)$
17. $xy = 4ik + 2i + 2k + 1$
18. $xy = 2(2ik + i + k) + 1$
19. xy is odd
20. It is not the case that both xy and $x + y$ are even
21. Therefore, if y is odd, then it is not the case that both xy and $x + y$ are even
22. If it is not the case that both x and y are even, then it is not the case that both xy and $x + y$ are even
23. If both xy and $x + y$ are even, then both x and y are even.