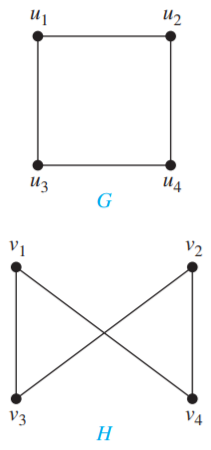
Section 13.3
Graph Isomorphism

Comparing Graphs

- In what way are the following two graphs the same?



Comparing Graphs

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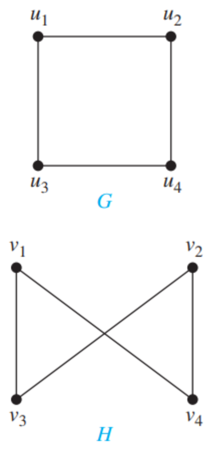
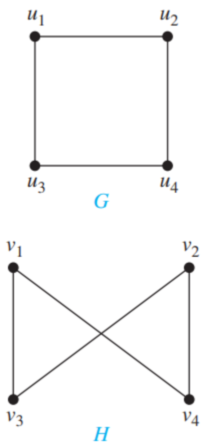


- They are both cycles of size 4

Comparing Graphs



$G = (V_G, E_G)$

$V_G = \{u_1, u_2, u_3, u_4\}$

$E_G = \{\{u_1, u_2\}, \{u_2, u_4\}, \{u_4, u_3\}, \{u_3, u_1\}\}$

$H = (V_H, E_H)$

$V_H = \{v_1, v_2, v_3, v_4\}$

$E_H = \{\{v_1, v_4\}, \{v_4, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$

It is possible to consider graph H as the result of replacing in G: u_1 with v_1, u_2 with v_4, u_3 with v_3, and u_4 with v_2

Undirected Graph Isomorphisms

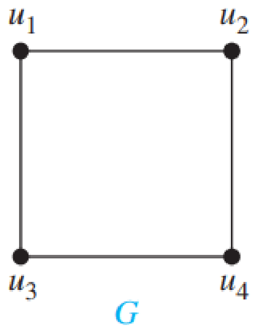
- Two undirected graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are isomorphic if there is a one-to-one correspondence (both one-to-one and onto) $f: V_G \to V_H$ such that for each $u \in V_G$ and $v \in V_G$, $\{u, v\} \in E_G$ if and only if $\{f(u), f(v)\} \in E_H$

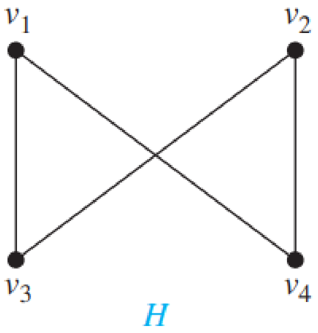
  - Such an f is called an isomorphism

Directed Graph Isomorphisms

- Two directed graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are isomorphic if there is a one-to-one correspondence (both one-to-one and onto) $f: V_G \to V_H$ such that for each $u \in V_G$ and $v \in V_G$, $(u, v) \in E_G$ if and only if $(f(u), f(v)) \in E_H$
  - Such an f is called an isomorphism

Graph Isomorphisms



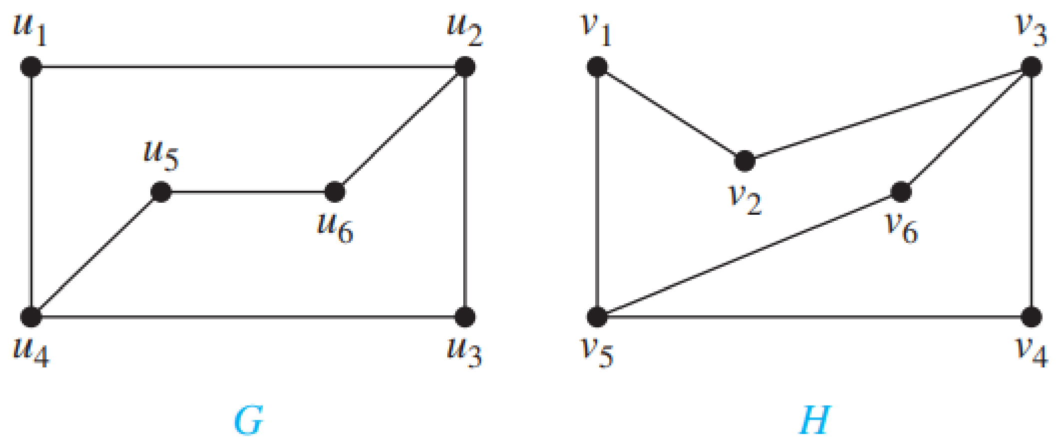


The graphs G and H are isomorphic via the isomorphism f where:

$$f(u_1) = v_1$$
$$f(u_2) = v_4$$
$$f(u_3) = v_3$$
$$f(u_4) = v_2$$

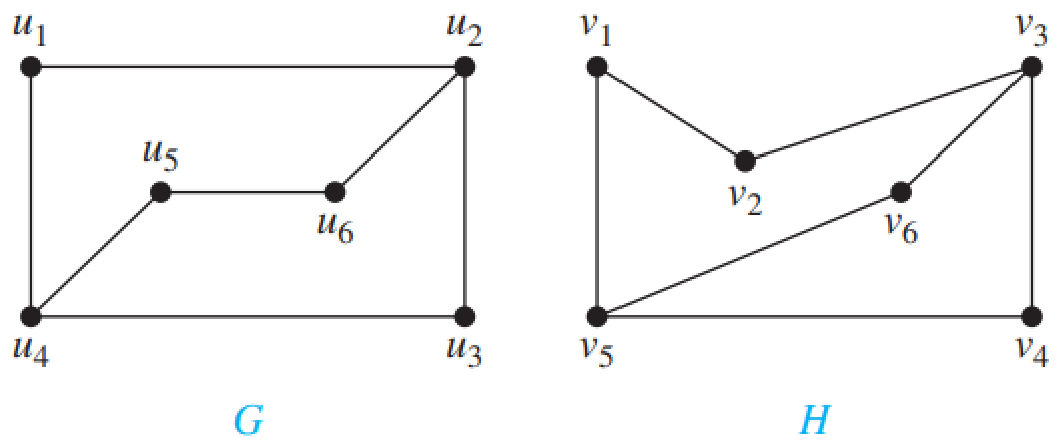
Graph Isomporhisms

- Example are the following two graphs isomorphic?



Graph Isomporhisms

- Example are the following two graphs isomorphic?



$$f(u_1) = v_6$$
$$f(u_2) = v_3$$
$$f(u_3) = v_4$$
$$f(u_4) = v_5$$
$$f(u_5) = v_1$$
$$f(u_6) = v_2$$

Properties Preserved Under Isomorphisms

- A property of a graph is preserved under isomorphisms if: whenever two graphs G and H are isomorphic, G has the property if and only if H has the property

Properties Preserved Under Isomorphisms

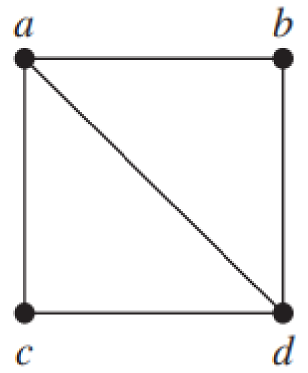
- Properties preserved under isomorphism include:
  - Number of vertices: $|V_G| = |V_H|$
  - Number of edges: $|E_G| = |E_H|$
  - The sum of the degrees of vertices: $\sum_{v \in V_G} \mathrm{degree}_G(v) = \sum_{v \in V_H} \mathrm{degree}_H(v)$
  - The existence of paths and cycles of particular lengths

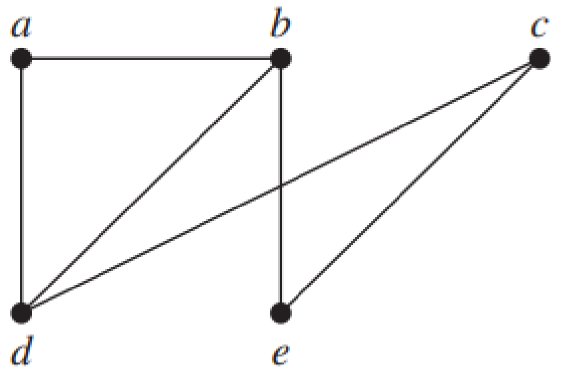
Determining Non-Isomorphism

- It can be difficult to determine if two graphs are isomorphic
- It can be easy to determine if two graphs are not isomorphic
  - Show that two graphs do not have the same property (for a property preserved under isomorphism)

Determining Non-Isomorphism

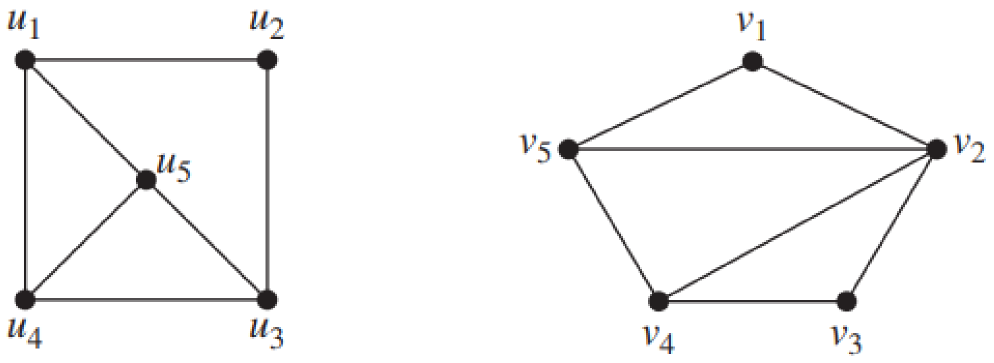
- Example: The following two graphs are not isomorphic because they have different a number of vertices (and also number edges)





Determining Non-Isomorphism

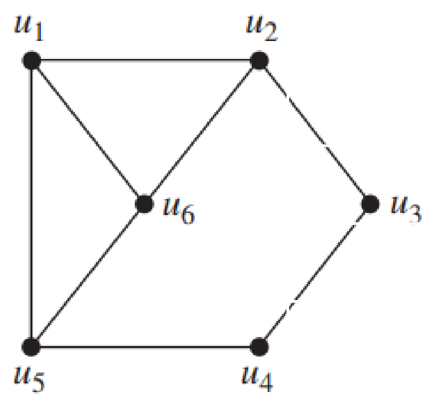
- Example: The following two graphs are not isomorphic because the graph on the left does not have a vertex of degree 4



Degree Sequences

- The degrees of the vertex of a graph can be put in a sorted sequence which is a property preserved under isomorphism

- Example: The following graph has a degree sequence of: 2, 2, 3, 3, 3, 3



$\text{degree}(u_1) = 3$

$\text{degree}(u_2) = 3$

$\text{degree}(u_3) = 2$

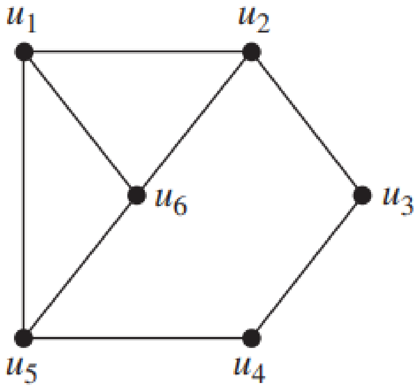
$\text{degree}(u_4) = 2$

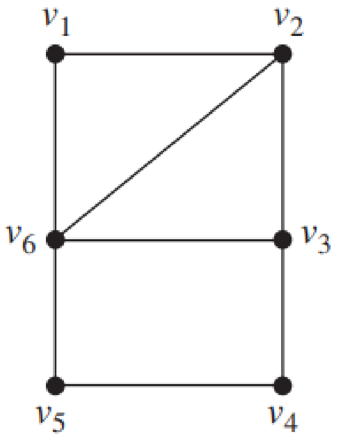
$\text{degree}(u_5) = 3$

$\text{degree}(u_6) = 3$

Determining Non-Isomorphism

- Example: The following two graphs are not isomorphic because they have different degree sequences



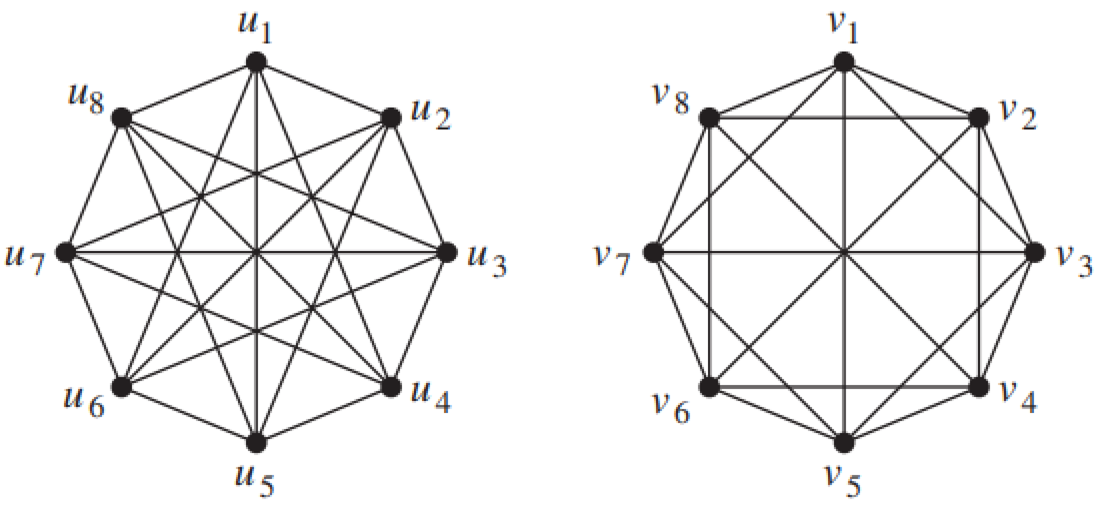


2, 2, 3, 3, 3, 3

2, 2, 2, 3, 3, 4

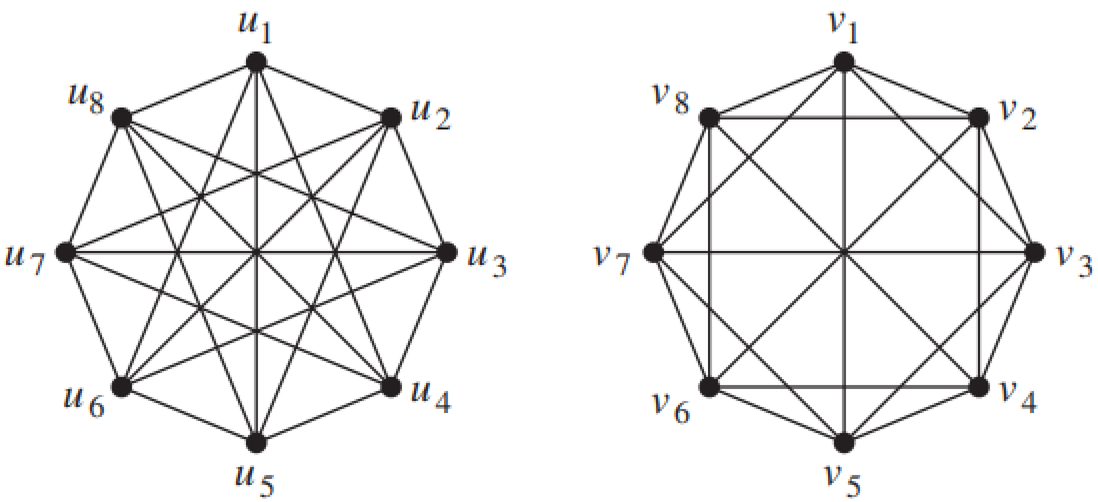
Determining Non-Isomorphism

- Example: Are the following two graphs isomorphic?



Determining Non-Isomorphism

- Example: Are the following two graphs isomorphic?



Triangles: u1,u2,u5; u1,u2,u6; u1,u4,u5; u1,u4,u8; u1,u5,u6; u1,u5,u8 u5 used in 4 triangles
v1,v2,v3; v1,v2,v8; v1,v3,v5; v1,v3,v7; v1,v5,v7; v1,v7,v8