Section 8.4
Mathematical Induction

Principle of Mathematical Induction

- Let the domain of discourse be the positive integers
- For a predicate P, we wish to prove $\forall n P(n)$
- To do this we first prove the predicate for the smallest positive integer, $P(1)$
- Then we prove that if the predicate is true for k, $P(k)$, then it is also true for $k + 1$:

$$P(k) \rightarrow P(k+1)$$

Principle of Mathematical Induction

- If we prove both $P(1)$ and $\forall k(P(k) \rightarrow P(k+1))$, then it must be the case that

$$\forall n P(n)$$

Because we have $P(1)$

and we have $P(2)$ because $P(1)$ and $P(1) \rightarrow P(2)$

and we have $P(3)$ because $P(2)$ and $P(2) \rightarrow P(3)$

and we have $P(4)$ because $P(3)$ and $P(3) \rightarrow P(4)$

$$\vdots$$

Proofs by Induction

- Example: Prove $\forall n P(n)$ by mathematical induction on the positive integers where

$$P(n) \text{ is } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1. Base case: Prove $P(1)$

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

   and $P(k+1)$ is $\sum_{i=1}^{k+1} i \ = \frac{(k+1)(k+2)}{2}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ We assume this

   and $P(k+1)$ is $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ We assume this

   and $P(k+1)$ is $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ We must conclude this

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

Note that $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

and $P(k+1)$ is $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k} i = \boxed{1 + 2 + \cdots + k}$$

$$\sum_{i=1}^{k+1} i = \boxed{1 + 2 + \cdots + k} + (k+1)$$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
   1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Proofs by Induction

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    1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
    2. $\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
    1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
    2. $\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$
    3. $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
   1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
   2. $\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$
   3. $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$
   4. $\qquad = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
   1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
   2. $\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$
   3. $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$
   4. $= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$
   5. $= \frac{k^2+k}{2} + \frac{2k+2}{2}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
    1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
    2. $\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$
    3. $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$
    4. $= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$
    5. $= \frac{k^2+k}{2} + \frac{2k+2}{2}$
    6. $= \frac{k^2+3k+2}{2}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
   1. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
   2. $\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$
   3. $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$
   4. $= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$
   5. $= \frac{k^2+k}{2} + \frac{2k+2}{2}$
   6. $= \frac{k^2+3k+2}{2}$
   7. $= \frac{(k+1)(k+2)}{2}$

Proofs by Induction

- Example 2: Prove $\forall n P(n)$ by mathematical induction where

  $P(n)$ is "The sum of the first n odd positive integers is n^2"

$P(1)$ is $1 = 1^2$

$P(2)$ is $1 + 3 = 2^2$

$P(3)$ is $1 + 3 + 5 = 3^2$

$P(4)$ is $1 + 3 + 5 + 7 = 4^2$

Proofs by Induction

- Example 2: Prove $\forall n P(n)$ by mathematical induction on the positive integers where

$$P(n) \text{ is "The sum of the first } n \text{ odd positive integers is } n^2\text{"}$$

$P(1)$ is $1 = 1^2$

$P(2)$ is $1 + 3 = 2^2$

$P(3)$ is $1 + 3 + 5 = 3^2$

$P(4)$ is $1 + 3 + 5 + 7 = 4^2$

n	nth odd number
1	1
2	3
3	5
4	7
k	$2k - 1$

Proofs by Induction

- Example 2: Prove $\forall n P(n)$ by mathematical induction on the positive integers where

$P(n)$ is "The sum of the first n odd positive integers is n^2"

1. Base case: Prove $P(1)$

$$1 = 1^2$$

Proofs by Induction

2. Induction step: Prove $P(k) \to P(k+1)$

   Note that $P(k)$ is $1 + 3 + \cdots + (2k-1) = k^2$

   and $P(k+1)$ is $1 + 3 + \cdots + (2k-1) + (2k+1) = (k+1)^2$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $1 + 3 + \cdots + (2k-1) = k^2$ We assume this

   and $P(k+1)$ is $1 + 3 + \cdots + (2k-1) + (2k+1) = (k+1)^2$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $1 + 3 + \cdots + (2k-1) = k^2$

   and $P(k+1)$ is $1 + 3 + \cdots + (2k-1) + (2k+1) = (k+1)^2$

We must conclude this

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\boxed{1 + 3 + \cdots + (2k-1)} = k^2$

   and $P(k+1)$ is $\boxed{1 + 3 + \cdots + (2k-1)} + (2k+1) = (k+1)^2$

Proofs by Induction

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Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

    1. Assume $1 + 3 + \cdots + (2k - 1) = k^2$
    2. $1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

    1. Assume $1 + 3 + \cdots + (2k-1) = k^2$
    2. $1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1)$
    3. $\quad = (k+1)(k+1)$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   1. Assume $1 + 3 + \cdots + (2k - 1) = k^2$
   2. $1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$
   3. $= (k + 1)(k + 1)$
   4. $= (k + 1)^2$

Induction on the Natural Numbers

- If the domain of discourse changes from the positive integers $\{1, 2, 3, \cdots\}$ to the natural numbers $\{0, 1, 2, \cdots\}$, then to prove

$$\forall n P(n)$$

by induction, we must start with the smallest natural number. So we prove

$$P(0)$$

and we still prove

$$P(k) \to P(k+1)$$

Proofs by Induction

- Example 3: Prove $\forall n P(n)$ by mathematical induction on the natural numbers where

$$P(n) \text{ is } \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

1. Base case: Prove $P(0)$

$$\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^1 - 1$$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

   and $P(k+1)$ is $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ We assume this

   and $P(k+1)$ is $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ We assume this

   and $P(k+1)$ is $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$ We must conclude this

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

   and $P(k+1)$ is $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

$$\sum_{i=0}^{k} 2^i = 2^0 + 2^1 + \cdots + 2^k$$

$$\sum_{i=0}^{k+1} 2^i = 2^0 + 2^1 + \cdots + 2^k + 2^{k+1}$$

Proofs by Induction

2. Induction step: Prove $P(k) \to P(k+1)$

   Note that $P(k)$ is $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

   and $P(k+1)$ is $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

$$\sum_{i=0}^{k} 2^i = \boxed{2^0 + 2^1 + \cdots + 2^k}$$

$$\sum_{i=0}^{k+1} 2^i = \boxed{2^0 + 2^1 + \cdots + 2^k} + 2^{k+1}$$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

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Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   1. Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$
   2. $\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

    1. Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$
    2. $\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$
    3. $\sum_{i=0}^{k+1} 2^i = 2^{k+1} - 1 + 2^{k+1}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   1. Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$
   2. $\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$
   3. $\sum_{i=0}^{k+1} 2^i = 2^{k+1} - 1 + 2^{k+1}$
   4. $= 2^{k+1} + 2^{k+1} - 1$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   1. Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$
   2. $\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$
   3. $\sum_{i=0}^{k+1} 2^i = 2^{k+1} - 1 + 2^{k+1}$
   4. $= 2^{k+1} + 2^{k+1} - 1$
   5. $= 2 \cdot 2^{k+1} - 1$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

    1. Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$
    2. $\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$
    3. $\sum_{i=0}^{k+1} 2^i = 2^{k+1} - 1 + 2^{k+1}$
    4. $= 2^{k+1} + 2^{k+1} - 1$
    5. $= 2 \cdot 2^{k+1} - 1$
    6. $= 2^{k+2} - 1$

Proofs by Induction

- Example 4: Prove $\forall n P(n)$ by mathematical induction on the natural numbers where

$P(n)$ is $\sum_{j=0}^{n} ar^j = ar^0 + ar^1 + \cdots + ar^n = \frac{ar^{n+1}-a}{r-1}$ when $r \neq 1$

1. Base case: Prove $P(0)$

$$\sum_{j=0}^{0} ar^j = a = \frac{a(r-1)}{r-1} = \frac{ar-a}{r-1} = \frac{ar^{0+1}-a}{r-1}$$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$

   and $P(k+1)$ is $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+2}-a}{r-1}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

Note that $P(k)$ is $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$ We assume this

and $P(k+1)$ is $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+2}-a}{r-1}$ We must conclude this

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   Note that $P(k)$ is $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$

   and $P(k+1)$ is $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+2}-a}{r-1}$

$$\sum_{j=0}^{k} ar^j = ar^0 + ar^1 + \cdots + ar^k$$

$$\sum_{j=0}^{k+1} ar^j = ar^0 + ar^1 + \cdots + ar^k + ar^{k+1}$$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

Note that $P(k)$ is $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$

and $P(k+1)$ is $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+2}-a}{r-1}$

$$\sum_{j=0}^{k} ar^j = \boxed{ar^0 + ar^1 + \cdots + ar^k}$$

$$\sum_{j=0}^{k+1} ar^j = \boxed{ar^0 + ar^1 + \cdots + ar^k} + ar^{k+1}$$

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Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
    1. Assume $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$
    2. $\sum_{j=0}^{k} ar^j + ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
   1. Assume $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$
   2. $\sum_{j=0}^{k} ar^j + ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
   3. $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$
    1. Assume $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$
    2. $\sum_{j=0}^{k} ar^j + ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
    3. $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
    4. $= \frac{ar^{k+1}-a}{r-1} + \frac{(r-1)\left(ar^{k+1}\right)}{r-1}$

Proofs by Induction

2. Induction step: Prove $P(k) \to P(k+1)$
    1. Assume $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$
    2. $\sum_{j=0}^{k} ar^j + ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
    3. $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
    4. $= \frac{ar^{k+1}-a}{r-1} + \frac{(r-1)\left(ar^{k+1}\right)}{r-1}$
    5. $= \frac{ar^{k+1}-a}{r-1} + \frac{rar^{k+1}-ar^{k+1}}{r-1}$

Proofs by Induction

2. Induction step: Prove $P(k) \rightarrow P(k+1)$

   1. Assume $\sum_{j=0}^{k} ar^j = \frac{ar^{k+1}-a}{r-1}$
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   3. $\sum_{j=0}^{k+1} ar^j = \frac{ar^{k+1}-a}{r-1} + ar^{k+1}$
   4. $= \frac{ar^{k+1}-a}{r-1} + \frac{(r-1)\left(ar^{k+1}\right)}{r-1}$
   5. $= \frac{ar^{k+1}-a}{r-1} + \frac{rar^{k+1}-ar^{k+1}}{r-1}$
   6. $= \frac{ar^{k+2}-a}{r-1}$