Section 8.2
Recurrence Relations

Recurrence Relations

- Note that we can give rules for defining a sequence $\{a_n\}_{n \in \boldsymbol{N}}$ where a_n is defined in terms of elements that precede it in the sequence

Recurrence Relations

- Example: The sequence $0, 1, 2, 3, \ldots$ can be described by the rule:

$$a_n = 1 + a_{n-1}$$

The first element of the sequence, a_0, is not given a value by this rule, so another rule is needed to define it:

$$a_0 = 0$$

Recurrence Relations

- Note that if a different value for a_0 is given then a different sequence is defined:

$$a_0 = 5$$
$$a_n = 1 + a_{n-1}$$

$$5, 6, 7, 8, \ldots$$

Recurrence Relations for Arithmetic Progressions

- Example: Compute a_3 for the arithmetic progression:

$$a_0 = 2$$

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- Example: Compute a_3 for the arithmetic progression:

$$a_0 = 2$$

$$a_n = a_{n-1} + 3$$

$$\begin{aligned} a_3 &= a_2 + 3 \\ &= a_1 + 3 + 3 \\ &= a_0 + 3 + 3 + 3 \\ &= 2 + 3 + 3 + 3 \\ &= 11 \end{aligned}$$

Recurrence Relations for Geometric Progressions

- A geometric progression can be defined by a recurrence relation

$$c, cr, cr^2, \cdots cr^n, \cdots$$

$$a_0 = c$$
$$a_n = a_{n-1} r$$

Recurrence Relations for Geometric Progressions

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$$a_n = a_{n-1} \cdot 5$$

$$\begin{aligned} a_3 &= a_2 \cdot 5 \\ &= a_1 \cdot 5 \cdot 5 \\ &= a_0 \cdot 5 \cdot 5 \cdot 5 \\ &= 4 \cdot 5 \cdot 5 \cdot 5 \end{aligned}$$

Recurrence Relations for Geometric Progressions

- Example: Compute a_3 for the geometric progression:

$$a_0 = 4$$

$$a_n = a_{n-1} \cdot 5$$

$$\begin{aligned} a_3 &= a_2 \cdot 5 \\ &= a_1 \cdot 5 \cdot 5 \\ &= a_0 \cdot 5 \cdot 5 \cdot 5 \\ &= 4 \cdot 5 \cdot 5 \cdot 5 \\ &= 500 \end{aligned}$$

The Fibonacci Sequence

- The Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

  is a sequence where each item is the sum of the two previous items in the sequence

The Fibonacci Sequence

- The Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

can be described by a recurrence relation:

$$f_0 = 0$$

$$f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2}$$

The Fibonacci Sequence

- Example: Compute f_4 of the Fibonacci sequence

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$$\begin{aligned} f_4 &= f_3 + f_2 \\ &= f_2 + f_1 + f_2 \end{aligned}$$

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\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2
\end{aligned}
$$

The Fibonacci Sequence

- Example: Compute f_4 of the Fibonacci sequence

$$\begin{aligned} f_4 &= f_3 + f_2 \\ &= f_2 + f_1 + f_2 \\ &= f_1 + f_0 + f_1 + f_2 \\ &= 1 + f_0 + f_1 + f_2 \end{aligned}$$

The Fibonacci Sequence

- Example: Compute f_4 of the Fibonacci sequence

$$
\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2
\end{aligned}
$$

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\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2 \\
&= 1 + 0 + 1 + f_2
\end{aligned}
$$

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\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2 \\
&= 1 + 0 + 1 + f_2 \\
&= 1 + 0 + 1 + f_1 + f_0
\end{aligned}
$$

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\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2 \\
&= 1 + 0 + 1 + f_2 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + f_1 + f_0
\end{aligned}
$$

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\begin{aligned}
f_4 &= f_3 + f_2 \\
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&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2 \\
&= 1 + 0 + 1 + f_2 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + 1 + f_0
\end{aligned}
$$

The Fibonacci Sequence

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\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2 \\
&= 1 + 0 + 1 + f_2 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + 1 + f_0 \\
&= 1 + 0 + 1 + 1 + 0
\end{aligned}
$$

The Fibonacci Sequence

- Example: Compute f_4 of the Fibonacci sequence

$$
\begin{aligned}
f_4 &= f_3 + f_2 \\
&= f_2 + f_1 + f_2 \\
&= f_1 + f_0 + f_1 + f_2 \\
&= 1 + f_0 + f_1 + f_2 \\
&= 1 + 0 + f_1 + f_2 \\
&= 1 + 0 + 1 + f_2 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + f_1 + f_0 \\
&= 1 + 0 + 1 + 1 + f_0 \\
&= 1 + 0 + 1 + 1 + 0 \\
&= 3
\end{aligned}
$$

Mutual Recurrence Relations

- Recurrence relations can be defined in terms of each other
- Example: A population of spotted owls can be divided into 3 groups: juveniles, subadults, and adults. Juveniles are the offspring of adult owls, transition to subadults, and then become adults which can reproduce. Unfortunately, not all owls survive to become adults. The dynamics of an owl population can be described as three groups of recurrence relations where index n describes the passage of time

$$j_n = 0.33 \cdot a_{n-1}$$

$$s_n = 0.60 \cdot j_{n-1}$$

$$a_n = 0.71 \cdot s_{n-1} + 0.94 \cdot a_{n-1}$$

Mutual Recurrence Relations

- Given the spotted owl recurrence relations, how does an initial population of 100 adult owls change over 3 units of time?

$$j_n = 0.33 \cdot a_{n-1}$$

$$s_n = 0.60 \cdot j_{n-1}$$

$$a_n = 0.71 \cdot s_{n-1} + 0.94 \cdot a_{n-1}$$

n	j_n	s_n	a_n
0	0	0	100
1	33	0	94
2	31	20	88
3	29	19	97