Section 4.1
Functions

Definition of a Function

- Let A and B be nonempty sets. A function f assigns each member of A to exactly one member of B
  - f is a function (mapping, transformation) from A to B

- $f: A \to B$ means f is a function from A to B

- The notation $f(a)$ denotes the member of set B assigned to a

Example

- Let $f: \{3, 4\} \rightarrow \{5, 10, 15\}$ where:

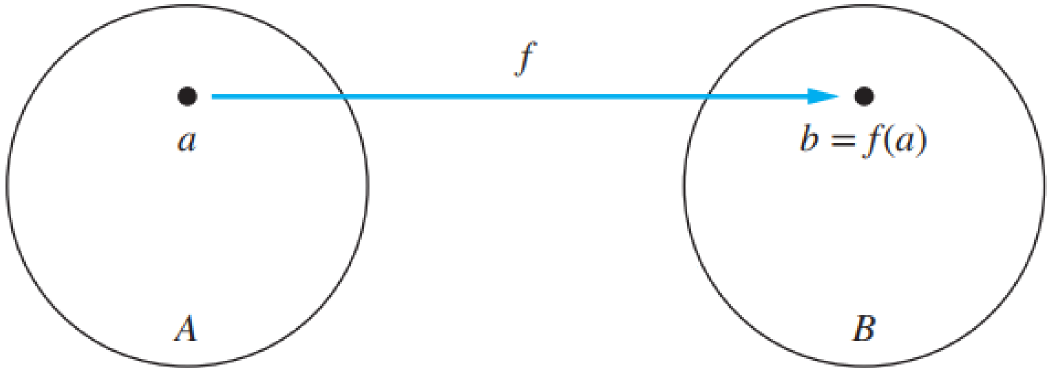
$$f(3) = 15$$

$$f(4) = 5$$

Domain and Co-domain of a Function

- When $f: A \to B$,
  - f maps A to B
  - A is the domain of f
  - B is the co-domain of f (also called the target of f)

- If we represent sets A and B with Venn diagrams, then we can think of a function $f: A \to B$ as the set of arrows from elements of A to elements of B such that each element of A having exactly one arrow starting from it and ending at an element of B



Images and Preimages

- When $f: A \to B$, and $f(a) = b$
  - b is the image of a
  - a is the preimage of b
  - The range (or image) of f is the set $\{f(a) | \, a \in A\}$

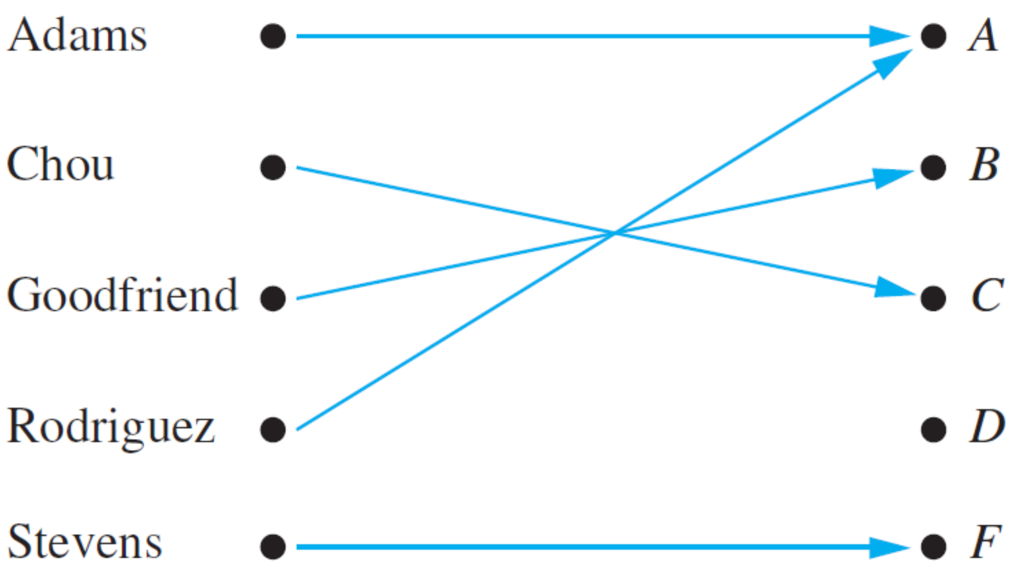
Example Co-domain vs. Range

- Let $f : \boldsymbol{Z} \rightarrow \boldsymbol{N}$ be a function from the integers to the natural numbers where: $f(x) = x^2$

  - The co-domain (target) of f is $\boldsymbol{N}$
  - The range of f is $\{0, 1, 4, 9, 16, 25, \ldots\}$

Example: Co-domain vs. Range

- Let G be a function assigning grades to students



- The domain of G is {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- The co-domain (target) of G is $\{\mathrm{A}, B, C, D, F\}$
- The range of G is $\{\mathrm{A}, B, C, F\}$
- The range of a function is always a subset of the function's co-domain (target)

Specifying a Function

- Often, a function can be specified by using a formula:

$$f: \boldsymbol{N} \rightarrow \boldsymbol{N}$$
$$f(x) = x + 1$$

$$g: \boldsymbol{Z} \rightarrow \boldsymbol{R}$$
$$g(x) = \sin(x)/2$$

$$h: \boldsymbol{N} \times \boldsymbol{Z} \rightarrow \boldsymbol{R}$$
$$h(x, y) = f(x) + g(y)$$

Specifying a Function

- A function can also be specified by using rules:

$$f: \boldsymbol{Z} \to \boldsymbol{Z}$$

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Specifying a Function

- A function matches each element of its domain to an element in its co-domain
- A function can be described as a set of ordered pairs

$$f: \boldsymbol{N} \rightarrow \boldsymbol{N}$$

$$f(x) = x^2$$

$$\{(0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), \cdots\}$$

$$\{(a, b) \mid a \in \boldsymbol{N}, b \in \boldsymbol{N}, \text{and } b = f(a)\}$$

Function Equality

- Two functions f and g are <u>equal</u> if:
  - f and g have the same domain
  - f and g have the same co-domain (target)
  - $f(x) = g(x)$ for every x in their domain

Images of Sets

- Let f be a function from set A to set B
- If $S \subseteq A$, then

$$f(S) = \{f(a) | \; a \in S\}$$

- $f(S)$ is called the image of S under f

- Note that the term "image" can also be used when a function is applied to a single member of the domain:
  - When $a \in A$, $f(a)$ is the image of a

Section 4.2
The Floor and Ceiling Functions

The Floor Function

- The floor function takes a real number and returns the largest integer that is less than or equal to the argument
- $floor: \boldsymbol{R} \rightarrow \boldsymbol{Z}$
- $floor(x)$ is also written as $\lfloor x \rfloor$

The Floor Function

- Examples

$$\lfloor 3.1 \rfloor = 3$$

$$\lfloor 3.8 \rfloor = 3$$

$$\lfloor 3 \rfloor = 3$$

$$\lfloor -3.5 \rfloor = -4$$

The Ceiling Function

- The ceiling function takes a real number and returns the smallest integer that is greater than or equal to the argument
- $ceil: \boldsymbol{R} \rightarrow \boldsymbol{Z}$
- $ceil(x)$ is also written as $\lceil x \rceil$

The Ceiling Function

- Examples

$$\lceil 3.1 \rceil = 4$$

$$\lceil 3.8 \rceil = 4$$

$$\lceil 3 \rceil = 3$$

$$\lceil -3.5 \rceil = -3$$

A Proof about the Ceiling Function

- If x is a real number, then $\lceil x \rceil < x + 1$
- Proof: by cases

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4. $\lceil x \rceil = x$
5. $\lceil x \rceil < x + 1$
6. If x is an integer, then $\lceil x \rceil < x + 1$

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- Proof: by cases
  1. Assume x is a real number
  2. There are two cases: x is an integer or x is not an integer
  3. Case 1: Assume x is an integer
  4. $\lceil x \rceil = x$
  5. $\lceil x \rceil < x + 1$
  6. If x is an integer, then $\lceil x \rceil < x + 1$
  7. Case 2: Assume that x is not an integer

A Proof about the Ceiling Function

- If x is a real number, then $\lceil x \rceil < x + 1$
- Proof: by cases

1. Assume x is a real number
2. There are two cases: x is an integer or x is not an integer
3. Case 1: Assume x is an integer
4. $\lceil x \rceil = x$
5. $\lceil x \rceil < x + 1$
6. If x is an integer, then $\lceil x \rceil < x + 1$
7. Case 2: Assume that x is not an integer
8. $x = n + d$ where n is an integer and d is a real number where $0 < d < 1$

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  11. If x is not an integer, then $\lceil x \rceil < x + 1$

A Proof about the Ceiling Function

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- Proof: by cases

1. Assume x is a real number
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12. If x is a real number, then $\lceil x \rceil < x + 1$