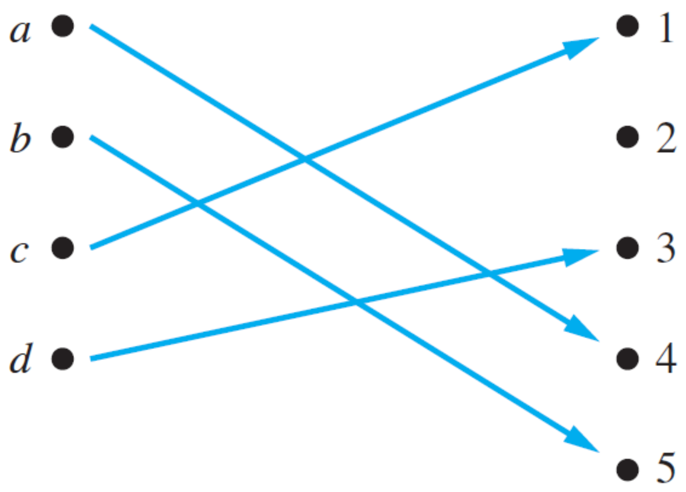
Section 4.3
Properties of Functions

One-to-One Functions

- A function is <u>one-to-one</u> if whenever $f(a) = f(b)$, then $a = b$
- One-to-one functions are also called <u>injective</u>
- Example:



If f is a one-to-one function that matches rabbits to rabbit-holes, then every rabbit-hole has at most one rabbit. (All rabbits are lonely)

One-to-One Functions

- Example: Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $f(x) = x^2$. Is f one-to-one?

One-to-One Functions

- Example: Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $f(x) = x^2$. Is f one-to-one?

- No, because there are two integers that are mapped to 1:

$$f(1) = f(-1) = 1$$

One-to-One Functions

- Example: Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $f(x) = x + 1$. Is f one-to-one?

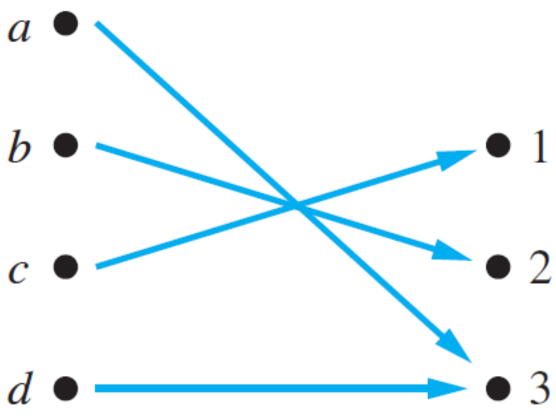
One-to-One Functions

- Example: Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $f(x) = x + 1$. Is f one-to-one?

- Yes, because if $f(x) = f(y)$, then $x + 1 = y + 1$ and therefore $x = y$

Onto Functions

- Let f be a function from set A to set B.
- f is onto if for each $b \in B$, there is at least one $a \in A$ such that $f(a) = b$
- Onto functions are also called surjective
- Example:



If f is an onto function that matches rabbits to rabbit-holes, then every rabbit-hole has at least one rabbit. (There are no empty rabbit-holes)

Onto Functions

- Example: Let $f: \boldsymbol{N} \to \boldsymbol{N}$ such that $f(x) = x + 1$. Is f onto?

Onto Functions

- Example: Let $f: \boldsymbol{N} \rightarrow \boldsymbol{N}$ such that $f(x) = x + 1$. Is f onto?

- No. There is no positive natural number n such that $f(n) = 0$

Onto Functions

- Example: Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $f(x) = x + 1$. Is f onto?

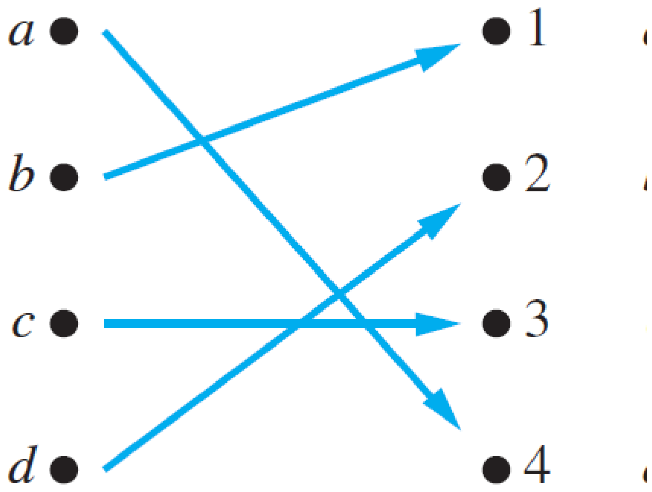
Onto Functions

- Example: Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $f(x) = x + 1$. Is f onto?

- Yes. Consider any $n \in \boldsymbol{Z}$. Since n is an integer, $n - 1$ is also an integer and $f(n - 1) = n$

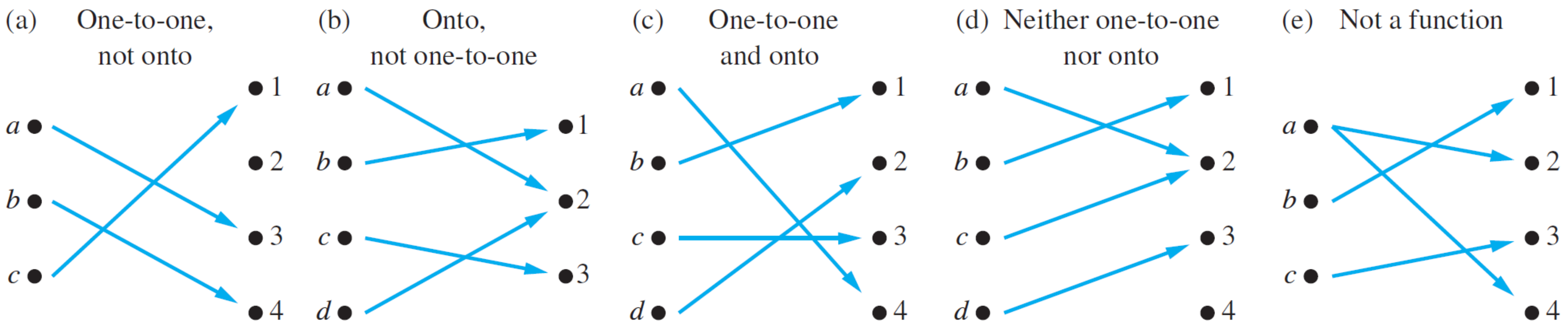
One-to-one Correspondence

- If a function f is both one-to-one and onto, then f is a one-to-one correspondence
- One-to-one correspondences are also called bijective
- Example:



If f is a one-to-one correspondence that matches rabbits to rabbit-holes, then every rabbit-hole has exactly one rabbit.

Comparison



(a) One-to-one, not onto
a
b
c
1
2
3
4
(b) Onto, not one-to-one
a
b
c
d
1
2
3
(c) One-to-one and onto
a
b
c
d
1
2
3
4
(d) Neither one-to-one nor onto
a
b
c
d
1
2
3
4
(e) Not a function
a
b
c
1
2
3
4

Summary

Let $f: A \to B$ be a function from A to B

- Show that f is one-to-one by showing that if $f(a_1) = f(a_2)$ where $a_1, a_2 \in A$, then $a_1 = a_2$
- Show that f is NOT one-to-one by showing $a_1, a_2 \in A$, $a_1 \neq a_2$ and $f(a_1) = f(a_2)$

- Show that f is onto by showing that for each $b \in B$, there is an $a \in A$ such that $f(a) = b$
- Show that f is NOT onto by showing that there is a $b \in B$, such that for each $a \in A$, $f(a) \neq b$

Examples

Let $f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$ where

$$f(a) = 4$$
$$f(b) = 5$$
$$f(c) = 1$$
$$f(d) = 3$$

Is f one-to-one (injective)?

Examples

Let $f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$ where

$f(a) = 4$

$f(b) = 5$

$f(c) = 1$

$f(d) = 3$

Is f one-to-one (injective)?

Yes

Examples

Let $f: \{a, b, c, d\} \rightarrow \{1, 2, 3\}$ where

$$f(a) = 3$$
$$f(b) = 2$$
$$f(c) = 1$$
$$f(d) = 3$$

Is f onto (surjective)?

Examples

Let $f: \{a, b, c, d\} \to \{1, 2, 3\}$ where

$f(a) = 3$

$f(b) = 2$

$f(c) = 1$

$f(d) = 3$

Is f onto (surjective)?

Yes

Examples

Let $f: \boldsymbol{R} \to \boldsymbol{R}$ where for any pair of real numbers x and y:

$$x < y \to f(x) < f(y)$$

Is f one-to-one (injective)?

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

1. Assume $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

1. Assume $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$
2. Assume that it is not the case that $x = y$

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

1. Assume $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$
2. Assume that it is not the case that $x = y$
3. There are two cases: $x < y$ and $y < x$

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

1. Assume $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$
2. Assume that it is not the case that $x = y$
3. There are two cases: $x < y$ and $y < x$
4. Without loss of generalization, assume $x < y$

Examples

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5. $f(x) < f(y)$

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4. Without loss of generalization, assume $x < y$
5. $f(x) < f(y)$
6. It is not the case that $f(x) = f(y)$

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

1. Assume $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$
2. Assume that it is not the case that $x = y$
3. There are two cases: $x < y$ and $y < x$
4. Without loss of generalization, assume $x < y$
5. $f(x) < f(y)$
6. It is not the case that $f(x) = f(y)$
7. If it is not the case that $x = y$ then it is not the case that $f(x) = f(y)$

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

1. Assume $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$
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7. If it is not the case that $x = y$ then it is not the case that $f(x) = f(y)$
8. If $f(x) = f(y)$ then $x = y$

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

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8. If $f(x) = f(y)$ then $x = y$
9. f is injective

Examples

Yes. Recall that for f to be injective, $f(x) = f(y) \rightarrow x = y$

Proof by contrapositive:

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4. Without loss of generalization, assume $x < y$
5. $f(x) < f(y)$
6. It is not the case that $f(x) = f(y)$
7. If it is not the case that $x = y$ then it is not the case that $f(x) = f(y)$
8. If $f(x) = f(y)$ then $x = y$
9. f is injective
10. If $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $x < y \rightarrow f(x) < f(y)$, then f is injective

Examples

Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ where $f(x) = x^2$

Is f onto (surjective)?

Examples

Let $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ where $f(x) = x^2$

Is f onto (surjective)?

No. There is no integer x such that $f(x) = 2$

Examples

Is the floor function $\lfloor \cdot \rfloor : \boldsymbol{R} \to \boldsymbol{Z}$ one-to-one (injective)? Is it onto (surjective)?

Examples

Is the floor function $\lfloor \cdot \rfloor : \boldsymbol{R} \to \boldsymbol{Z}$ one-to-one (injective)? Is it onto (surjective)?

It is not one-to-one because $\lfloor 1.1 \rfloor = \lfloor 1.2 \rfloor = 1$

It is onto. For any $n \in \boldsymbol{Z}$, $\lfloor n \rfloor = n$

Examples

Let $f: \boldsymbol{Z}^+ \rightarrow \boldsymbol{R}^+$ where $f(x) = \frac{1}{x}$

Is f one-to-one (injective)? Is it onto (surjective)?

Examples

Let $f: \boldsymbol{Z}^{+} \rightarrow \boldsymbol{R}^{+}$ where $f(x) = \frac{1}{x}$

Is f one-to-one (injective)? Is it onto (surjective)?

f is one-to-one. If $f(x) = f(y)$, then $\frac{1}{x} = \frac{1}{y}$ and hence $x = y$

f is not onto. There is no positive integer x such that $f(x) = 2$

Examples

Let $f: \boldsymbol{R}^+ \rightarrow \boldsymbol{R}^+$ where $f(x) = \frac{1}{x}$

Is f one-to-one (injective)? Is it onto (surjective)?

Examples

Let $f: \boldsymbol{R}^+ \rightarrow \boldsymbol{R}^+$ where $f(x) = \frac{1}{x}$

Is f one-to-one (injective)? Is it onto (surjective)?

f is one-to-one. If $f(x) = f(y)$, then $\frac{1}{x} = \frac{1}{y}$ and hence $x = y$

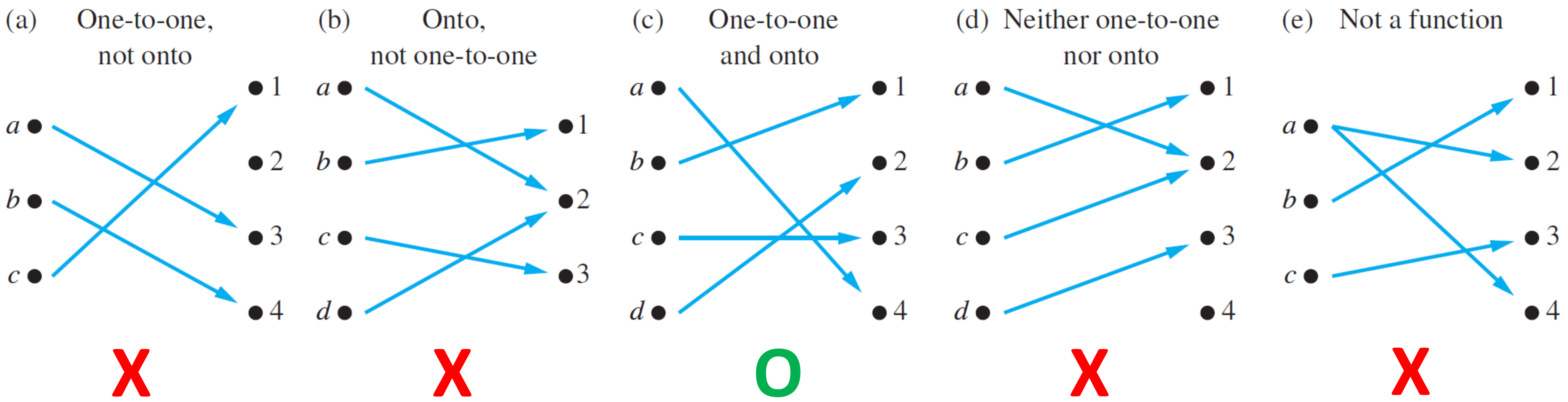
f is onto. For any $n \in \boldsymbol{R}^+$, it is the case that $\frac{1}{n} \in \boldsymbol{R}^+$. So $f\left(\frac{1}{n}\right) = \frac{1}{\frac{1}{n}} = n$

Inverse Functions

- Sometimes, we want to undo or reverse a function by using another function.
- Let $f: A \rightarrow B$ be a one-to-one correspondence
- If whenever $f(a) = b$, $g(b) = a$:
  - $g(f(a)) = a$
  - Function g is the inverse of function f
  - Since g is a function, $g: B \rightarrow A$
  - In general, we denote the inverse of f as $f^{-1}: B \rightarrow A$

Inverse Functions

- Not all functions have inverses



- Only one-to-one correspondences have inverses

Invertible Functions

- Because one-to-one correspondences have inverses, they are sometimes called invertible

Invertible Functions

- Sometimes being invertible depends on the domain and codomain of a function
- $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ where $f(x) = x + 1$ is invertible
- $f: \boldsymbol{Z}^{+} \rightarrow \boldsymbol{Z}^{+}$where $f(x) = x + 1$ is NOT invertible
- $g: \boldsymbol{R}^{+} \rightarrow \boldsymbol{R}^{+}$where $\mathrm{g}(x) = x^{2}$ is invertible
- $g: \boldsymbol{R} \rightarrow \boldsymbol{R}$ where $\mathrm{g}(x) = x^{2}$ is NOT invertible