Chapter 2
Proofs

Even and Odd Integers

- An integer x is <u>even</u> if there is an integer k such that $x = 2k$

- An integer x is <u>odd</u> if there is an integer k such that $x = 2k + 1$

Rational Numbers

- A number r is rational if there are integers x and y such that:

  - $y \neq 0$ and
  - $r = \frac{x}{y}$

  - Examples: $\frac{2}{3}, \frac{0}{5}, \frac{4}{1}, \frac{-1}{2}$

Divides

- An integer x divides an integer y if:
  - $x \neq 0$ and
  - For some integer k, $y = x \cdot k$

- If x divides y
  - x is a divisor or factor of y
  - y is a multiple of x

Prime and Composite Numbers

- An integer n is <u>prime</u> if and only if $n > 1$ and the only positive integers that divide n are 1 and n

- An integer n is <u>composite</u> if and only if $n > 1$ and there is an integer m such that: $1 < m < n$ and m divides n

Inequalities

- If x and c are real numbers, then exactly one of the following is true:
  - $x < c$
  - $x = c$
  - $x > c$
- In addition:
  - $x \geq c$ if $x > c$ or $x = c$
  - $x \leq c$ if $x < c$ or $x = c$

Inequalities

- If it is not the case that $x < c$, then $x = c$ or $x > c$

- If it is not the case that $x > c$, then $x = c$ or $x < c$

- If it is not the case that $x = c$, then $x < c$ or $x > c$

Inequalities

- If $x < c$, then $x \leq c$

- If $x > c$, then $x \geq c$

Positive and Negative Numbers

- A real number x is <u>positive</u> if and only if $x > 0$
- A real number x is <u>negative</u> if and only if $x < 0$

Theorems and Proofs

- A theorem is a statement that can be proven true

- A proof is a sequence of statements where each statement:
  - is an assumption, or
  - is a previously proven statement, or
  - logically follows from previous statements in the proof

Things you may assume in proofs

- The rules of algebra such as commutativity, associativity, distributivity
- The sum of two integers is an integer
- The difference of two integers is an integer
- The product of two integers is an integer
- Each integer is either odd or even
- If x is an integer, there is no integer between x and $x + 1$
- Any two real numbers x and y are comparable: either $x = y$, $x < y$, or $x > y$
- If x is a real number, then $x^2 \geq 0$

Proofs of Implications

- Assume the antecedent and derive the consequent
- Example: If x is an odd integer, then $3x + 1$ is an even integer
  - Proof:
    1. Assume x is an odd integer

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    2. $x = 2i + 1$ for some integer i

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    3. $3x + 1 = 3(2i + 1) + 1$

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    4. $ 6i + 3 + 1$

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4. $ 6i + 3 + 1$
5. $ 6i + 4$

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2. $x = 2i + 1$ for some integer i
3. $3x + 1 = 3(2i + 1) + 1$
4. $ = 6i + 3 + 1$
5. $ = 6i + 4$
6. $ = 2(3i + 2)$ where $3i + 2$ is an integer

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- Example: If x is an odd integer, then $3x + 1$ is an even integer
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2. $x = 2i + 1$ for some integer i
3. $3x + 1 = 3(2i + 1) + 1$
4. $6i + 3 + 1$
5. $6i + 4$
6. $2(3i + 2)$ where $3i + 2$ is an integer
7. $3x + 1$ is an even integer

Proofs of Implications

- Assume the antecedent and derive the consequent
- Example: If x is an odd integer, then $3x + 1$ is an even integer
  - Proof:

1. Assume x is an odd integer
2. $x = 2i + 1$ for some integer i
3. $3x + 1 = 3(2i + 1) + 1$
4. $\qquad\qquad 6i + 3 + 1$
5. $\qquad\qquad 6i + 4$
6. $\qquad\qquad 2(3i + 2)$ where $3i + 2$ is an integer
7. $3x + 1$ is an even integer
8. If x is an odd integer, then $3x + 1$ is an even integer

Do Not Work Backwards

- Do NOT start from a conclusion and then end at an assumption

1. $2 = 3$
2. $2 \cdot 0 = 3 \cdot 0$
3. $0 = 0$
4. if $0 = 0$ then $2 = 3$

This is an incorrect proof.

Theorems that are Universal or Existential Statements

- Some theorems apply to all elements of a set and therefore are implicitly universally quantified

- Example: The sum of two positive real numbers is greater than the average of the two numbers
- The statement is about <u>any</u> two positive real numbers

$$\forall x \forall y \big((x > 0 \land \ y > 0) \rightarrow (x + y > (x + y)/2) \big)$$

Theorems that are Universal or Existential Statements

- Some theorems are about the existence of a particular object

- Example: There is an integer that is equal to its square

$$\exists x\left(x = x^2\right)$$

Proofs by Exhaustion for Universal Statements

- Some universal statements can be proven by considering a small number of specific values

Proofs by Exhaustion for Universal Statements

- Example: $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$

Proofs by Exhaustion for Universal Statements

- Example: $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$
    2. There are four possible values for n: 1, 2, 3, 4

Proofs by Exhaustion for Universal Statements

- Example: $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$
    2. There are four possible values for n: 1, 2, 3, 4
    3. If $n = 1$ then $(1+1)^3 = 2^3 = 8 \geq 3 = 3^1$

Proofs by Exhaustion for Universal Statements

- Example: $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$
    2. There are four possible values for n: 1, 2, 3, 4
    3. If $n = 1$ then $(1+1)^3 = 2^3 = 8 \geq 3 = 3^1$
    4. If $n = 2$ then $(2+1)^3 = 3^3 = 27 \geq 9 = 3^2$

Proofs by Exhaustion for Universal Statements

- Example: $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$
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    3. If $n = 1$ then $(1+1)^3 = 2^3 = 8 \geq 3 = 3^1$
    4. If $n = 2$ then $(2+1)^3 = 3^3 = 27 \geq 9 = 3^2$
    5. If $n = 3$ then $(3+1)^3 = 4^3 = 64 \geq 27 = 3^3$

Proofs by Exhaustion for Universal Statements

- Example: $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$
    2. There are four possible values for n: 1, 2, 3, 4
    3. If $n = 1$ then $(1+1)^3 = 2^3 = 8 \geq 3 = 3^1$
    4. If $n = 2$ then $(2+1)^3 = 3^3 = 27 \geq 9 = 3^2$
    5. If $n = 3$ then $(3+1)^3 = 4^3 = 64 \geq 27 = 3^3$
    6. If $n = 4$ then $(4+1)^3 = 5^3 = 125 \geq 81 = 3^4$

Proofs by Exhaustion for Universal Statements

- Example: $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$
  - Proof:
    1. Assume n is a positive integer with $n \leq 4$
    2. There are four possible values for n: 1, 2, 3, 4
    3. If $n = 1$ then $(1+1)^3 = 2^3 = 8 \geq 3 = 3^1$
    4. If $n = 2$ then $(2+1)^3 = 3^3 = 27 \geq 9 = 3^2$
    5. If $n = 3$ then $(3+1)^3 = 4^3 = 64 \geq 27 = 3^3$
    6. If $n = 4$ then $(4+1)^3 = 5^3 = 125 \geq 81 = 3^4$
    7. $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$

Proofs by Generalization for Universal Statements

- When there are too many domain elements for a proof by exhaustion, a proof by generalization might be possible
- For a proof by generalization, start with an arbitrary object (usually a variable) with no assumptions other than the assumptions of the theorem

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
- Note that this statement is equivalent to an implication:

If a number is a positive integer, then it is less than or equal to its square

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
  - Proof:
    1. Assume x is a positive integer

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
  - Proof:
    1. Assume x is a positive integer
    2. $0 < x$

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
  - Proof:
    1. Assume x is a positive integer
    2. $0 < x$
    3. $1 \leq x$

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
  - Proof:
    1. Assume x is a positive integer
    2. $0 < x$
    3. $1 \leq x$
    4. $1 \cdot x \leq x \cdot x$ because x is positive

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
  - Proof:
    1. Assume x is a positive integer
    2. $0 < x$
    3. $1 \leq x$
    4. $1 \cdot x \leq x \cdot x$ because x is positive
    5. $x \leq x^2$

Proofs by Generalization for Universal Statements

- Example: Any positive integer is less than or equal to its square
  - Proof:
    1. Assume x is a positive integer
    2. $0 < x$
    3. $1 \leq x$
    4. $1 \cdot x \leq x \cdot x$ because x is positive
    5. $x \leq x^2$
    6. Any positive integer is less than or equal to its square

Counterexamples for Universal Statements

- Sometimes we are not sure if a statement is a theorem or not.
- If it is not a theorem, we won't be able to prove it.
- We can check if the statement is not a theorem by finding a counterexample for it.
- A counterexample for a universal statement is a value for which the statement is false

Counterexamples for Universal Statements

- Example: If n is an integer greater than 1, then $2^n < n^3$
- Although the statement is true if $2 \leq n \leq 9$, it is not true if $n = 10$:

$$2^{10} = 1024 \nless 1000 = 10^3$$

- So, $n = 10$ is a counterexample to the statement

Counterexamples for Universally Quantified Conditional Statements

- If a statement is a universally quantified implication

$$\forall x(P(x) \rightarrow Q(x))$$

Then a counterexample would be a value for x such that $P(x)$ is true but $Q(x)$ is false

- Example:

If x is an odd integer and $x > 4$, then $2x$ is odd

Counterexample: $x = 5$

Constructive Proofs of Existential Statements

- Existential statement:

$$\exists x P(x)$$

- A constructive proof of an existential statement produces or describes how to effectively produce a domain element that makes the statement true

Proofs of Existential Statements

- Example 1: There is a prime number that is the sum of two prime numbers
- Proof:
  1. 2, 3, and 5 are prime numbers
  2. 5 = 2 + 3

Proofs of Existential Statements

- Example 2: There is an integer that is 1 greater than the number of people on earth
- Proof: The finite number of people on earth can be counted. Let n be the number of people on earth. Then $n + 1$ is an integer that is 1 greater than the number of people on earth

Nonconstructive Proofs of Existential Statements

- Example: In my pocket, there is a piece of paper that correctly answers the question, "Does God exist?"
- Proof: The correct answer to the question is either "yes" or "no". In my pocket there are two pieces of paper. On one is written "yes"; on the other is written "no".

Disproving Existential Statements

- To disprove an existential statement, you must show that it is false. However, by De Morgan's law for quantifiers:

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

- Thus, to disprove $\exists x P(x)$, you must show $\neg P(x)$ for all domain elements x