Section 6.3
Directed Graphs, Paths and Cycles

Directed Graphs

- A directed graph $G = (V, E)$ consists of:
  - V, a non-empty set of vertices (or nodes)
  - E, a set of directed edges,
    - $E \subseteq V \times V$
    - A directed edge $(u, v) \in E$ connects a tail vertex (or initial vertex) u to a head vertex (or terminal) v.
    - The vertices connected by an edge are called its endpoints

Directed Graph Example

- Example: Let $G = (V, E)$ where:
  - $V = \{\text{San Franciso, Los Angeles, Denver, Chicago, Detroit, Washington, New York}\}$
  - $E = \{$(San Francisco, Los Angeles), (San Francisco, Denver) , (Los Angeles, Denver), (Denver, Chicago), (Chicago, Detroit), (Chicago, Washington), (Chicago, New York), (Detroit, New York)$\}$

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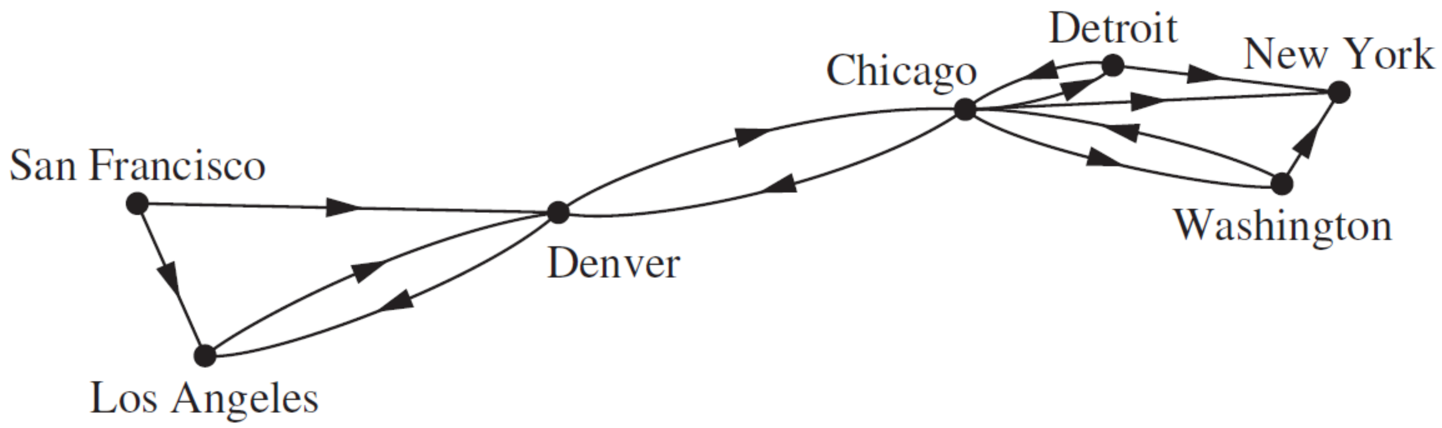


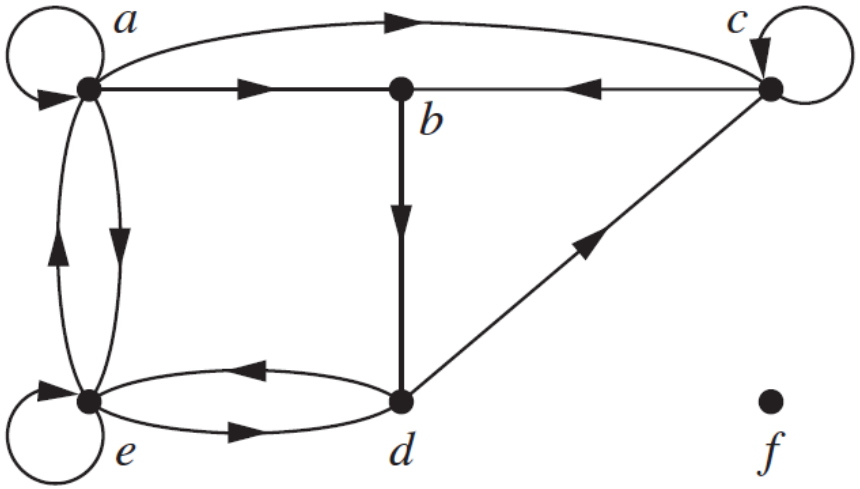
In-degree and Out-degree

- Let $G = (V, E)$ be a directed graph
  - The in-degree of a vertex $v \in V$ is the number of edges that have v as a terminal vertex
  - $\text{in}-\text{degree}(v) = |\{(u, v) \mid (u, v) \in E\}|$

  - The out-degree of a vertex $v \in V$ is the number of edges that have v as an initial vertex
  - $\text{out}-\text{degree}(v) = |\{(v, w) \mid (v, w) \in E\}|$

In-degree and Out-degree

- Example: What are the in-degrees and out-degrees of each vertex in the following graph?



$\text{in}-\text{degree}(a) = 2$ $\quad$ $\text{out}-\text{degree}(a) = 4$

$\text{in}-\text{degree}(b) = 2$ $\quad$ $\text{out}-\text{degree}(b) = 1$

$\text{in}-\text{degree}(c) = 3$ $\quad$ $\text{out}-\text{degree}(c) = 2$

$\text{in}-\text{degree}(d) = 2$ $\quad$ $\text{out}-\text{degree}(d) = 2$

$\text{in}-\text{degree}(e) = 3$ $\quad$ $\text{out}-\text{degree}(e) = 3$

$\text{in}-\text{degree}(f) = 0$ $\quad$ $\text{out}-\text{degree}(f) = 0$

Walks in Directed Graphs

- A walk in a directed graph $G = (V, E)$ from vertex $v_0 \in V$ to vertex $v_k \in V$ is a sequence of alternating vertices and edges beginning with v_0 and ending with v_k:

$$(v_0, (v_0, v_1), v_1(v_1, v_2)v_2, \dots v_k)$$

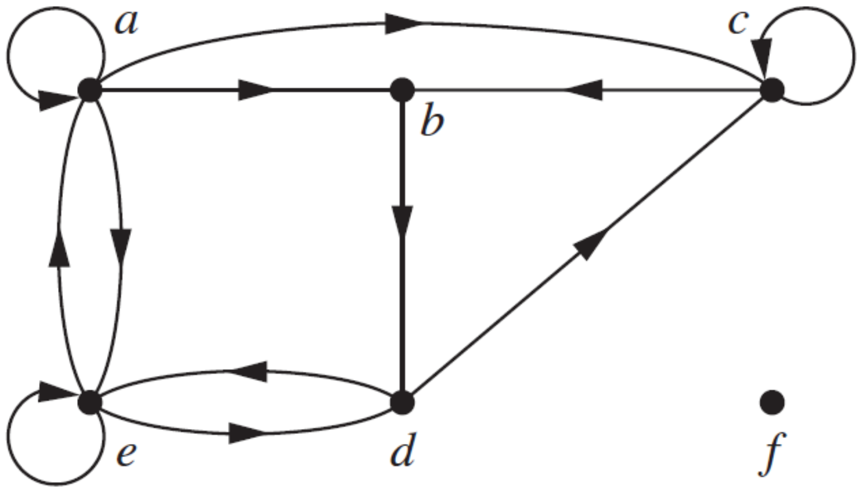
Where each edge (v_i, v_{i+1}) is flanked by vertices v_i and v_{i+1}

Walks in Directed Graphs

- The length of a walk is the number of edges in the walk.
- An open walk is a walk that starts and ends at different vertices
- A closed walk is a walk that starts and ends at the same vertex

Walks in Directed Graphs

- Example: In the following directed graph



  - $(a, (a, c), c, (c, b), b, (b, d), d)$ is an open walk of length 3
  - $(a, (a, b), b, (b, d), d, (d, e), e, (e, a), a)$ is a closed walk of length 4

Walks in Directed Graphs

- Since each edge in a walk is determined by its flanking vertices, a walk can be abbreviated by its sequence of vertices.

$$(v_0, (v_0, v_1), v_1, (v_1, v_2), v_2, \dots v_k)$$

$$(v_0, v_1, v_2, \dots v_k)$$

Trails and Paths

- A trail is a walk in which no edge occurs more than once
- A path is a walk in which no vertex occurs more than once

Trail and Path Examples

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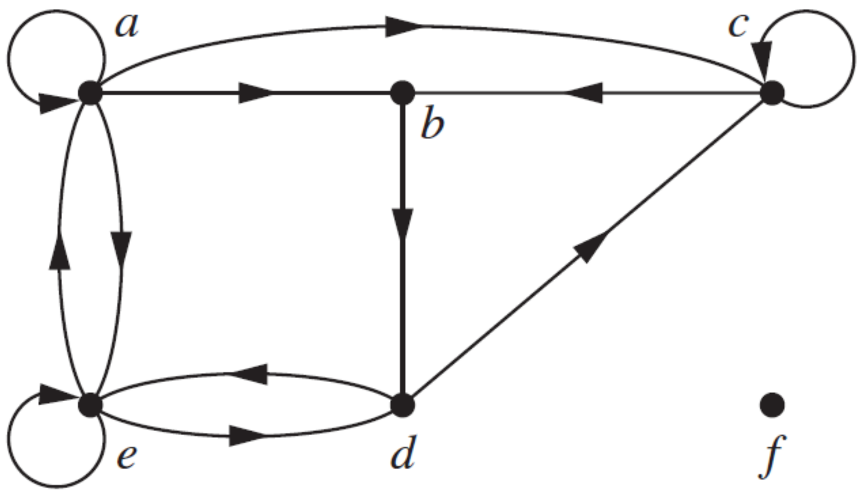
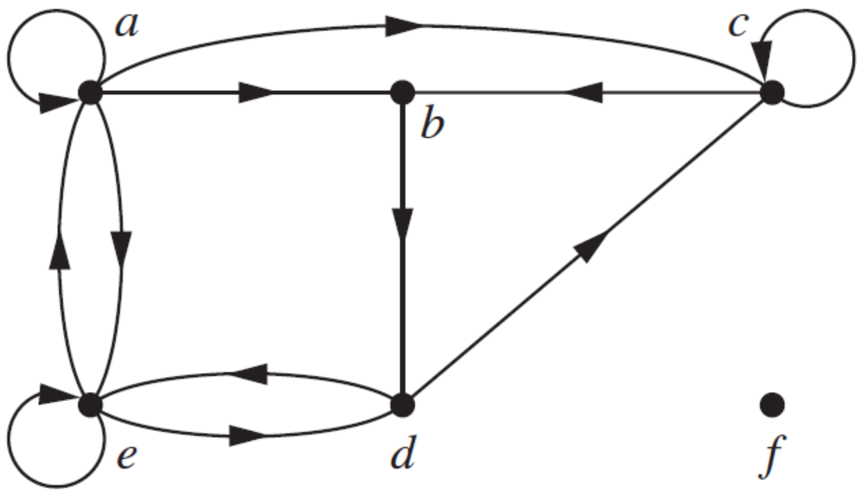
- A path is a walk in which no vertex occurs more than once



- (a, e, d, e) is a trail (but not a path)
- (a, b, d, c) is a path

Circuits and Cycles

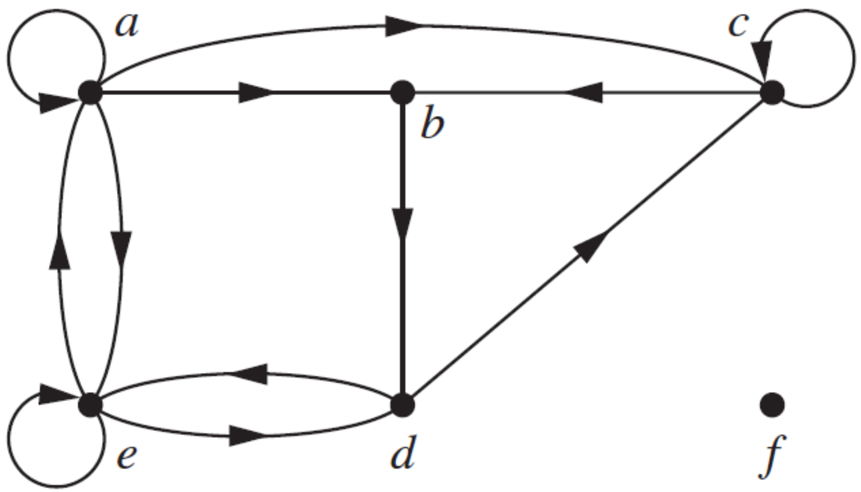
- A <u>circuit</u> is a closed trail
- A <u>cycle</u> is a circuit of length at least 1 in which no vertex occurs more than once except the first and last vertices



- (a, e, d, e, a) is a circuit (but not a cycle)
- (a, c, b, d, e, a) is a cycle

Neighbors

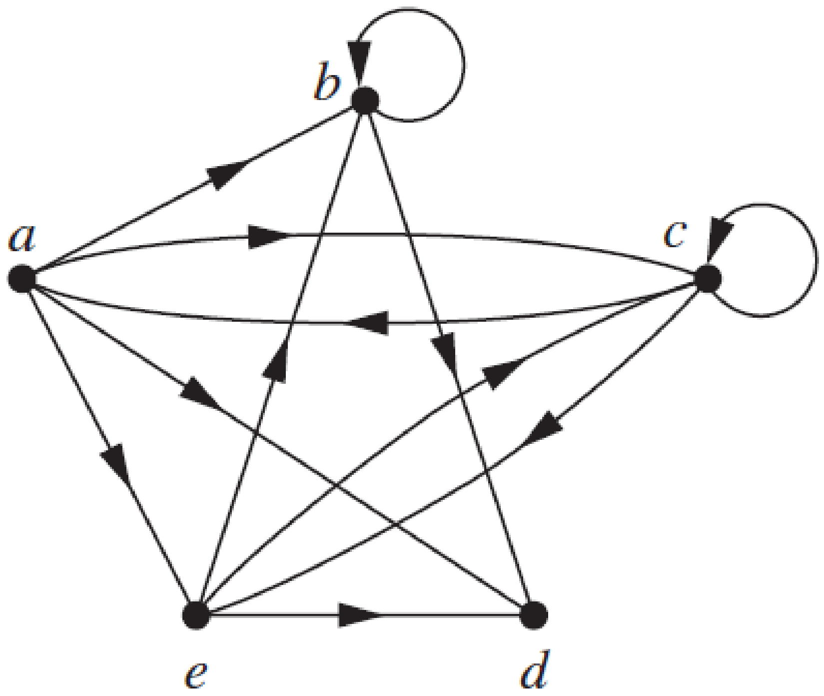
- Vertex v is an <u>out-neighbor</u> of vertex u in a directed graph if the graph has an edge (u, v)



- Vertex d has out-neighbors c and e

Paths

- Example: In the following graph, there is a path from vertex c to vertex b: c, a, b. However, there is no path from vertex b to vertex c



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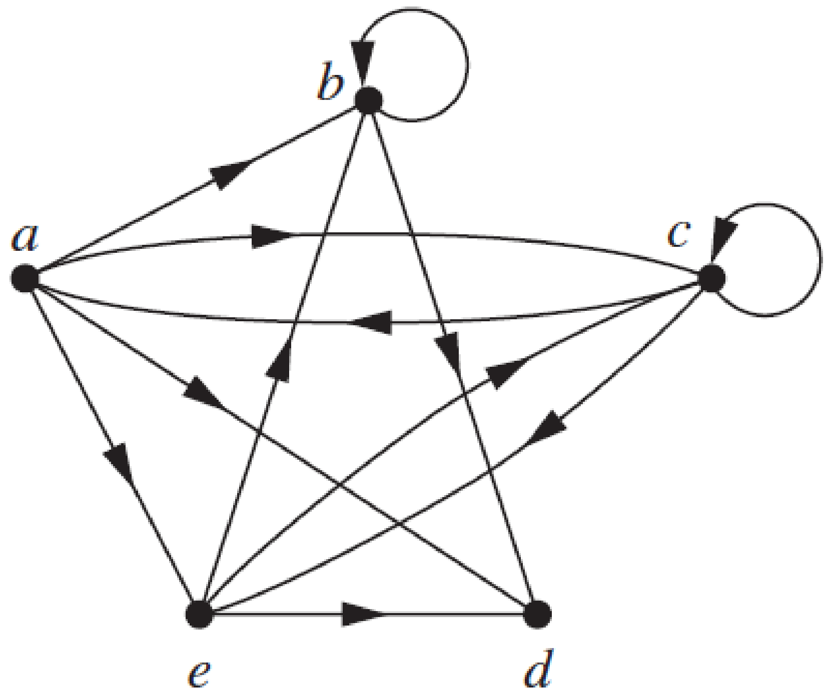

Repeatedly find out-neighbors starting with $\{c\}$ until: 1) the neighbors include b, or 2) the neighbors do not change

$$Neighbors(\{c\}) = \{a, c, e\}$$

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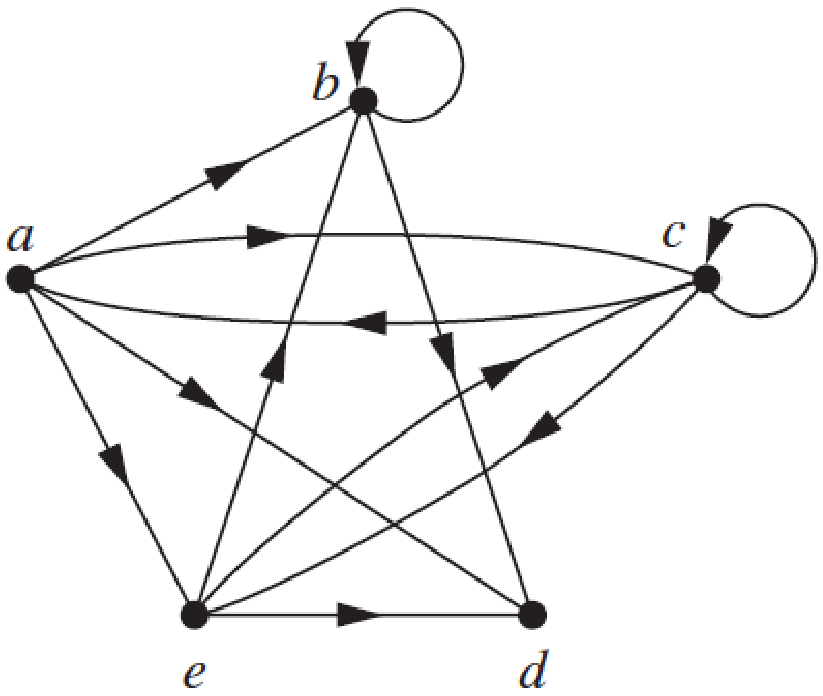

Repeatedly find out-neighbors starting with $\{b\}$ until: 1) the neighbors include c, or 2) the neighbors do not change

$$Neighbors(\{b\}) = \{b, d\}$$
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