Section 3.4
More Set Operations

Set Difference

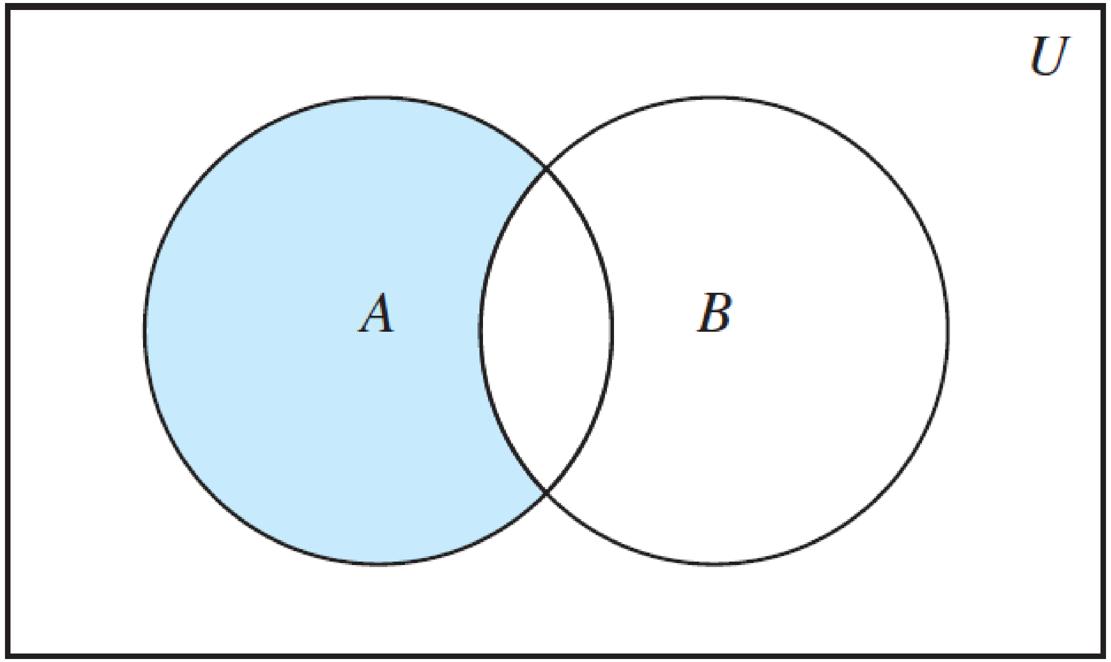
- If A and B are sets, then the difference of A and B is the set containing exactly the members of A that are not also members of B

$$A - B = \{x \mid x \in A \land x \notin B\}$$

$$\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$$

$$\{1, 2, 3\} - \{1, 3, 5\} = \{2\}$$

Set Difference



$A - B$ is shaded.

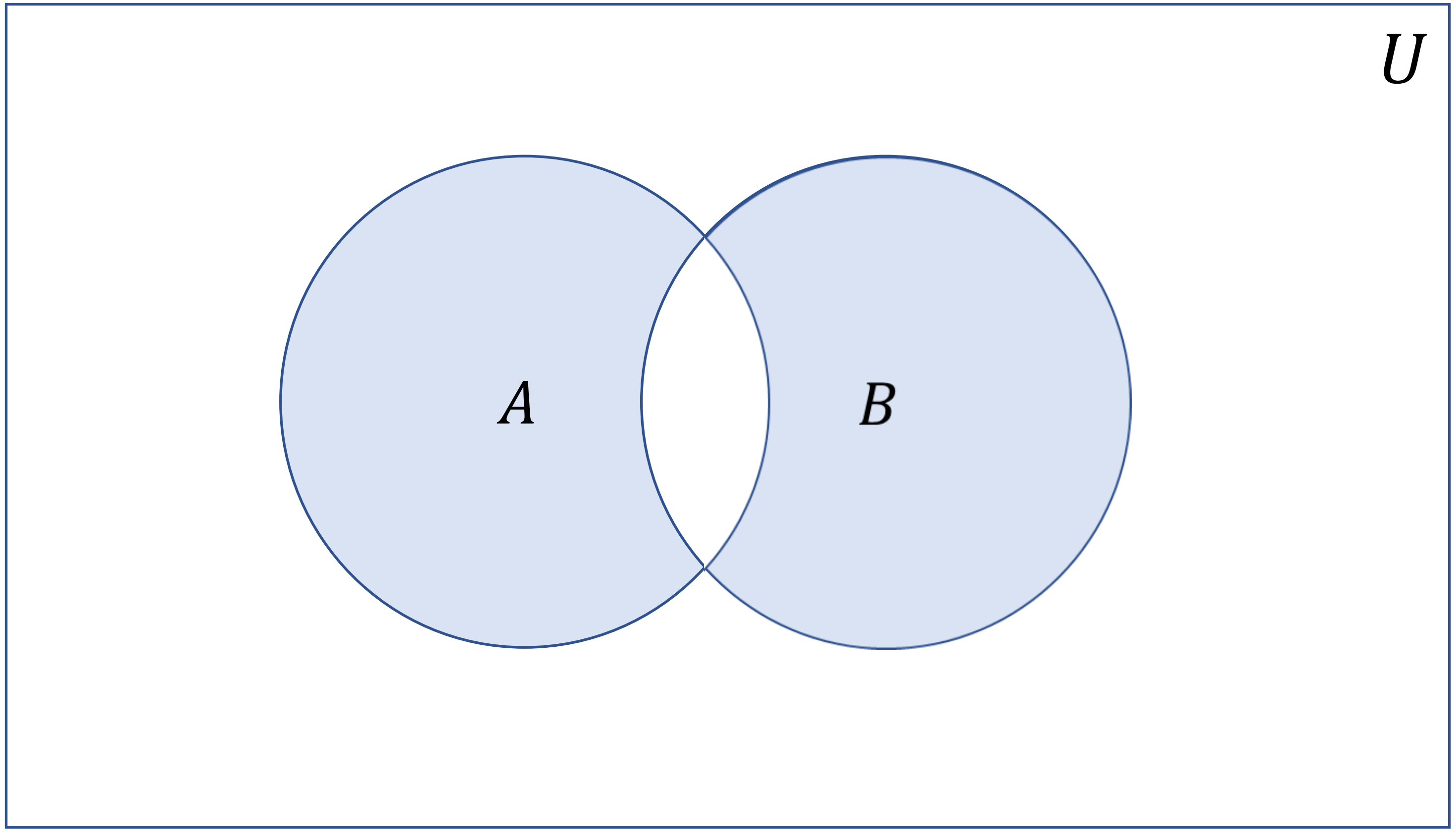
Symmetric Difference

- If A and B are sets, then the symmetric difference of A and B is the set containing exactly the members of A that are not also in B and the members of B that are not also in A

$$A \oplus B = (A - B) \cup (B - A)$$

$$\{1, 3, 5\} \oplus \{1, 2, 3\} = \{2,5\}$$

Symmetric Difference



$A \oplus B$ is shaded

Set Complement

- The complement of a set A is the set containing the members of the universal set, U, that are not also in A

$$\overline{A} = \{x \mid x \in U \wedge x \notin A\}$$

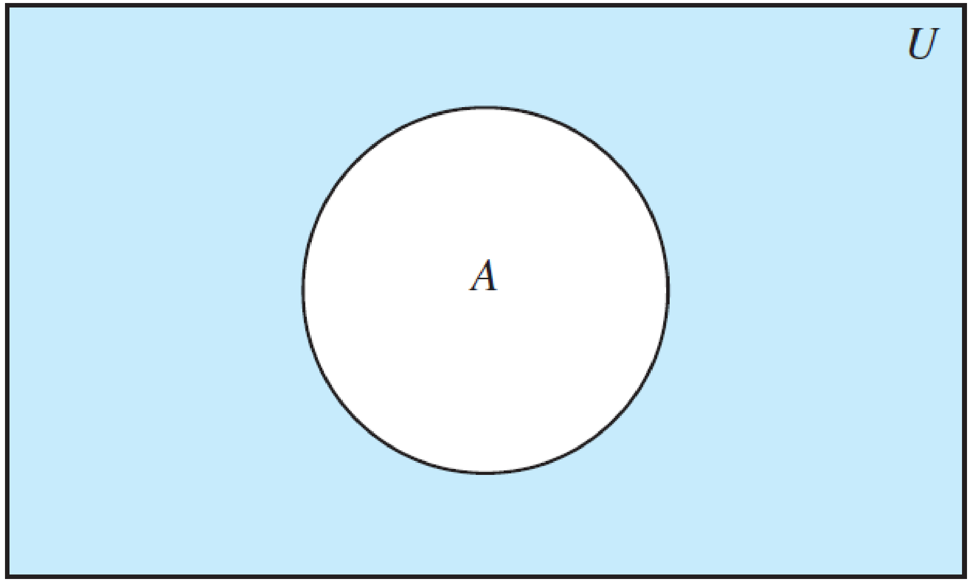
- $x \in \overline{A}$ if and only if $x \notin A$

Set Complement

- If the universal set is $\boldsymbol{Z}^{+}$ (all positive integers),
- and A is the set of positive integers greater than 10, $A = \{11, 12, 13, \dots\}$. Then:

$$\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Set Complement



$\overline{A}$ is shaded.

Examples

- The universe exactly contains the integers from 1 to 8, $U = \{1,2,3,4,5,6,7,8\}$
- $A = \{1, 4, 5, 7\}$
- $B = \{2, 4, 6, 7\}$
- $C = \{3, 5, 6, 7\}$

$$\begin{aligned}\overline{A} \cap \overline{B} &= \overline{\{1, 4, 5, 7\}} \cap \overline{\{2, 4, 6, 7\}} \\ &= \{2, 3, 6, 8\} \cap \{1, 3, 5, 8\} \\ &= \{3, 8\}\end{aligned}$$

$$\begin{aligned}\overline{A \cap B} &= \overline{\{1, 4, 5, 7\} \cap \{2, 4, 6, 7\}} \\ &= \overline{\{4, 7\}} \\ &= \{1, 2, 3, 5, 6, 8\}\end{aligned}$$

Examples

- The universe exactly contains the integers from 1 to 8, , $U = \{1,2,3,4,5,6,7,8\}$
- $A = \{1, 4, 5, 7\}$
- $B = \{2, 4, 6, 7\}$
- $C = \{3, 5, 6, 7\}$

$$\begin{aligned} \overline{A} - \overline{C} &= \overline{\{1, 4, 5, 7\}} - \overline{\{3, 5, 6, 7\}} \\ &= \{2, 3, 6, 8\} - \{1, 2, 4, 8\} \\ &= \{3, 6\} \end{aligned}$$

$$\begin{aligned} \overline{A - C} &= \overline{\{1, 4, 5, 7\} - \{3, 5, 6, 7\}} \\ &= \overline{\{1, 4\}} \\ &= \{2, 3, 5, 6, 7,8\} \end{aligned}$$

Section 3.5
Set Identities

Set Identities

- The set operations union, intersection and complement can be expressed using logical operations

  - $x \in A \cup B \leftrightarrow (x \in A \vee x \in B)$
  - $x \in A \cap B \leftrightarrow (x \in A \wedge x \in B)$
  - $x \in \overline{A} \leftrightarrow \neg(x \in A)$

  - $x \in U \leftrightarrow T$
  - $x \in \emptyset \leftrightarrow F$

De Morgan's Law for Set Intersection

$$
\begin{aligned}
x \in \overline{A \cap B} &\leftrightarrow \neg(x \in A \cap B) \\
&\leftrightarrow \neg(x \in A \land x \in B) \\
&\leftrightarrow \neg(x \in A\ ) \lor \neg(x \in B) \\
&\leftrightarrow x \in \overline{A} \lor x \in \overline{B} \\
&\leftrightarrow x \in \overline{A} \cup \overline{B}
\end{aligned}
$$

Therefore $\overline{A \cap B} = \overline{A} \cup \overline{B}$

De Morgan's Law for Set Union

$$
\begin{aligned}
x \in \overline{A \cup B} \ &\leftrightarrow \neg(x \in A \cup B) \\
&\leftrightarrow \neg(x \in A \vee x \in B) \\
&\leftrightarrow \neg(x \in A\ ) \wedge \neg(x \in B) \\
&\leftrightarrow x \in \overline{A} \wedge x \in \overline{B} \\
&\leftrightarrow x \in \overline{A} \cap \overline{B}
\end{aligned}
$$

Therefore $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Set Identity Laws

Set Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Double Complement law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan’s laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proving Sets are Equal Using Set Identities

- Example: Show that $\overline{A \cup (B \cap C)} = \left(\overline{C} \cup \overline{B}\right) \cap \overline{A}$

$\overline{A \cup (B \cap C)}$	$= \overline{A} \cap \overline{B \cap C}$	De Morgan
	$= \overline{A} \cap \left(\overline{B} \cup \overline{C}\right)$	De Morgan
	$= \left(\overline{B} \cup \overline{C}\right) \cap \overline{A}$	Commutativity of intersection
	$= \left(\overline{C} \cup \overline{B}\right) \cap \overline{A}$	Commutativity of union

Proving Sets are Equal Using Membership Tables

- Similar to truth tables
- Each column corresponds to a set
- The leftmost columns are for set variables
- Each row is for a possible combination of sets that an item can be a member of

Proving Sets are Equal Using Membership Tables

- Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Columns for $\overline{A \cap B}, \overline{A} \cup \overline{B}, A \cap B, \overline{A}, \overline{B}, A, B$
- Rows for the possible ways an item can be in or not in A and B
- Use 1 to indicate membership and 0 to indicate non-membership

Proving Sets are Equal Using Membership Tables

- Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

Since the columns for $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are the same, they are equal