Section 3.6
Cartesian Products

Unordered Sets

- Sets are unordered

$$\{1, 2, 3\} = \{2, 3, 1\}$$

Ordered N-Tuples

- Sometimes the order of items is important such as when we want to talk about the first, second, and third place finishers of a race

1st	Chris
2nd	Stacy
3rd	Sandy

- In this case, we can use an ordered triple:

$$(\text{Chris}, \text{Stacy}, \text{Sandy})$$

- Note that we use parentheses instead of the curly braces used for sets

Ordered N-Tuples

- In general, to establish an order of n items, we use an ordered n-tuple

$$(a_1, a_2, \cdots, a_n)$$

- Since the order matters:

$$(\text{Chris, Stacy, Sandy}) \neq (\text{Stacy, Sandy, Chris})$$

Ordered N-Tuples

- Unlike sets, repetition is allowed in ordered n-tuples

- Example: what are the different ways that some one can give you 7-cents using 3 coins?

  (penny, penny, nickel)

  (penny, nickel, penny)

  (nickel, penny, penny)

Ordered N-Tuples

- Two n-tuples are equal if they have the same items in the same order:

$$(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)$$

if and only if

$$\forall i (1 \leq i \leq n \rightarrow a_i = b_i)$$

Cartesian Product

- The cartesian product of two sets A and B, $A \times B$, is the set containing all of the ways that a member of A can be paired with a member of B.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

- Note that (a, b) is an ordered pair (2-tuple)

Cartesian Product

- What is {chocolate, vanilla, strawberry} × {ice cream, milkshake}?

- A table may be helpful:

	ice cream	milkshake
chocolate	(chocolate, ice cream)	(chocolate, milkshake)
vanilla	(vanilla, ice cream)	(vanilla,milkshake)
strawberry	(strawberry,ice cream)	(strawberry,milkshake)

Cartesian Product

- What is $\{\text{chocolate, vanilla, strawberry}\} \times \{\text{ice cream, milkshake}\}$?

$$\begin{aligned}&\{\text{chocolate, vanilla, strawberry}\} \times \{\text{ice cream, milkshake}\} \\ &= \{(\text{chocolate, ice cream}), (\text{chocolate, milkshake}), \\ &\quad (\text{vanilla, ice cream}), (\text{vanilla, milkshake}), \\ &\quad (\text{strawberry, ice cream}), (\text{strawberry, milkshake})\}\end{aligned}$$

Cartesian Product

- Any finite number of sets can be combined using the cartesian product:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots a_n) \mid a_1 \in A_1, a_2 \in A_2, \cdots a_n \in A_n\}$$

Cartesian Product

- Example

$$\{1,2\} \times \{a,b\} \times \{X\} \times \{3,c\} = \{(1,a,X,3), (1,a,X,c), (1,b,X,3), (1,b,X,c), (2,a,X,3), (2,a,X,c), (2,b,X,3), (2,b,X,c)\}$$

Cartesian Product

- The cartesian product of a set A with itself is abbreviated as A^2

$$A^2 = A \times A = \{(a, b) \mid a \in A \text{ and } b \in A\}$$

- In general:

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

Cartesian Product

- Cartesian products are not always finite. Recall that $\boldsymbol{N}$ is the set of natural numbers, $\{0, 1, 2, 3, \dots\}$

$$\boldsymbol{N} \times \boldsymbol{N}$$
$$= \{(0,0), (0,1), (0,2), \dots, (1,0), (1,1), (1,2), \dots (2,0), (2,1), (2,2), \dots, \dots\}$$

Sets of N-Tuples

- The cartesian product of sets A and B, $A \times B$, contains all possible pairs of values from A and B

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

- It is possible to have subsets of $A \times B$

Sets of N-Tuples

- Example: Let $A = \{a, b, c\}$ and $B = \{1, 2\}$
- $A \times B = \{(a, 1),\ (a, 2),\ (b, 1),\ (b, 2),\ (c, 1), (c, 2)\}$

$$\{(a, 1), (b, 1), (c, 1)\} \subseteq A \times B$$

$$\{a, c\} \times \{2\} \subseteq A \times B$$

Example 1

- What is $\emptyset \times \{x, y, z\}$?

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- $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

- $\emptyset \times \{x, y, z\} = \emptyset$

Example 2

- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

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  - $A \subseteq C$ means if $x \in A$ then $x \in C$
  - $B \subseteq D$ means if $x \in B$ then $x \in D$
  - $A \times B \subseteq C \times D$ means if $(x, y) \in A \times B$ then $(x, y) \in C \times D$
  - $(x, y) \in A \times B$ means $x \in A$ and $y \in B$
  - $(x, y) \in C \times D$ means $x \in C$ and $y \in D$

Example 2

- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

  First, try using a few sets

  - $A = \{x, y\}$
  - $B = \{1, 2\}$
  - $C = \{x, y, z\}$
  - $D = \{1, 2, 3\}$
  - $A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2)\}$
  - $C \times D = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3), (z, 1), (z, 2), (z, 3)\}$

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1. Assume $A \subseteq C$ and $B \subseteq D$

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1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$

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- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$
3. $x \in A$ and $y \in B$

Example 2

- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$
3. $x \in A$ and $y \in B$
4. $x \in C$ and $y \in D$ because $A \subseteq C$ and $B \subseteq D$

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1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$
3. $x \in A$ and $y \in B$
4. $x \in C$ and $y \in D$ because $A \subseteq C$ and $B \subseteq D$
5. $(x, y) \in C \times D$

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- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$
3. $x \in A$ and $y \in B$
4. $x \in C$ and $y \in D$ because $A \subseteq C$ and $B \subseteq D$
5. $(x, y) \in C \times D$
6. If $(x, y) \in A \times B$ then $(x, y) \in C \times D$

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- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$
3. $x \in A$ and $y \in B$
4. $x \in C$ and $y \in D$ because $A \subseteq C$ and $B \subseteq D$
5. $(x, y) \in C \times D$
6. If $(x, y) \in A \times B$ then $(x, y) \in C \times D$
7. $A \times B \subseteq C \times D$

Example 2

- Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

1. Assume $A \subseteq C$ and $B \subseteq D$
2. Assume $(x, y) \in A \times B$
3. $x \in A$ and $y \in B$
4. $x \in C$ and $y \in D$ because $A \subseteq C$ and $B \subseteq D$
5. $(x, y) \in C \times D$
6. If $(x, y) \in A \times B$ then $(x, y) \in C \times D$
7. $A \times B \subseteq C \times D$
8. If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$

Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

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Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$
    1. Assume A, B, and C are nonempty sets and $A \times B = A \times C$
    2. $a \in A$ for some a in A since A is not empty

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1. Assume A, B, and C are nonempty sets and $A \times B = A \times C$
2. $a \in A$ for some a in A since A is not empty
3. Assume $x \in B$

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1. Assume A, B, and C are nonempty sets and $A \times B = A \times C$
2. $a \in A$ for some a in A since A is not empty
3. Assume $x \in B$
4. $(a, x) \in A \times B$

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    1. Assume A, B, and C are nonempty sets and $A \times B = A \times C$
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    3. Assume $x \in B$
    4. $(a, x) \in A \times B$
    5. $(a, x) \in A \times C$

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  1. Assume A, B, and C are nonempty sets and $A \times B = A \times C$
  2. $a \in A$ for some a in A since A is not empty
  3. Assume $x \in B$
  4. $(a, x) \in A \times B$
  5. $(a, x) \in A \times C$
  6. $x \in C$

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- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

1. Assume A, B, and C are nonempty sets and $A \times B = A \times C$
2. $a \in A$ for some a in A since A is not empty
3. Assume $x \in B$
4. $(a, x) \in A \times B$
5. $(a, x) \in A \times C$
6. $x \in C$
7. If $x \in B$ then $x \in C$

Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

   8. Assume $x \in C$

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- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

8. Assume $x \in C$
9. $(a, x) \in A \times C$

Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

8. Assume $x \in C$
9. $(a, x) \in A \times C$
10. $(a, x) \in A \times B$

Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

8. Assume $x \in C$
9. $(a, x) \in A \times C$
10. $(a, x) \in A \times B$
11. $x \in B$

Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

8. Assume $x \in C$
9. $(a, x) \in A \times C$
10. $(a, x) \in A \times B$
11. $x \in B$
12. If $x \in C$ then $x \in B$

Example 3

- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

8. Assume $x \in C$
9. $(a, x) \in A \times C$
10. $(a, x) \in A \times B$
11. $x \in B$
12. If $x \in C$ then $x \in B$
13. $x \in B$ if and only if $x \in C$

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8. Assume $x \in C$
9. $(a, x) \in A \times C$
10. $(a, x) \in A \times B$
11. $x \in B$
12. If $x \in C$ then $x \in B$
13. $x \in B$ if and only if $x \in C$
14. $B = C$

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- Prove that if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$

8. Assume $x \in C$
9. $(a, x) \in A \times C$
10. $(a, x) \in A \times B$
11. $x \in B$
12. If $x \in C$ then $x \in B$
13. $x \in B$ if and only if $x \in C$
14. $B = C$
15. if A, B, and C are nonempty sets and $A \times B = A \times C$ then $B = C$