Homework Assignment 7
CS 2233
Section 001 and Section 002
Due: Friday, April 12

Problem 1. [10 points]
Complete all participation activities in zyBook sections 8.6-8.11

Problem 2. [20 points] Consider a proof by strong induction on the set $\{12, 13, 14, \dots\}$ of $\forall n\, P(n)$ where $P(n)$ is: n cents of postage can be formed by using only 3-cent stamps and 7-cent stamps

a. [5 points] For the base case, show that $P(12)$, $P(13)$, and $P(14)$ are true

12 cents of postage is 4 3-cent stamps
13 cents of postage is 2 3-cent stamps and 1 7-cent stamp
14 cents of postage is 2 7-cent stamps

b. [5 points] What is the induction hypothesis?

$P(12) \wedge P(13) \wedge \cdots \wedge P(k+2)$

c. [5 points] What do you need to prove for the inductive step?

If $P(12) \wedge P(13) \wedge \cdots \wedge P(k+2)$ then $P(k+3)$ or If any amount of postage from 12 to $k+2$ cents can be formed by using only 3-cent stamps and 7-cent stamps, then $k+3$ cents of postage can be formed from 3-cent stamps and 7-cent stamps

d. [5 points] Complete the inductive step for $k+3$ cents of postage

1.	Any amount of postage from 12 to $k+2$ cents can be formed by using only 3-cent stamps and 7-cent stamps	Induction hypothesis
2.	k cents of postage can be formed by using only 3-cent stamps and 7-cent stamps	
3.	$k+3$ cents of postage can be formed using only 3-cent stamps and 7-cent stamps by adding a 3-cent stamp to the stamps that formed k cents of postage	

Problem 3. [5 points] Prove by using strong induction on the positive integers $\forall n P(n)$ where $P(n)$ is: The positive integer n can be expressed as the sum of different powers of 2

For example, $19 = 16 + 2 + 1 = 2^4 + 2^1 + 2^0$

Hint: For the inductive step, separately consider the cases where $k+1$ is even and odd. When $k+1$ is even, $(k+1)/2$ is an integer.

1. Base case: $1 = 2^0$

2. Induction step: Prove if $P(1)$, $P(2)$, …, and $P(k)$, then $P(k+1)$:

For all $i \leq k$, i is the sum of different powers of 2	Assumption
Consider $k+1$. It is either even or odd	
Case 1: $k+1$ is even	
$(k+1)/2$ is an integer and $(k+1)/2 \leq k$	
$(k+1)/2 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_j}$ where each $a_1, a_2, \dots, a_j$ are different	By the assumption
$(k+1) = 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_j+1}$ where each $a_1+1, a_2+1, \dots, a_j+1$ are different	
If $k+1$ is even, then it is the sum of different powers of 2	

Case 1: $k+1$ is odd

$k/2$ is an integer and $k/2 \leq k$

$k/2 = 2^{a_1}+2^{a_2}+\cdots+2^{a_j}$ where each $a_1, a_2, \dots, a_j$ are different By the assumption

$k = 2^{a_1+1}+2^{a_2+1}+\cdots+2^{a_j+1}$ where each $a_1+1, a_2+1, \dots, a_j+1$ are different

$k+1 = 2^{a_1+1}+2^{a_2+1}+\cdots+2^{a_j+1}+2^0$ where each $a_1+1, a_2+1, \dots, a_j+1$ and 0 are different

If $k+1$ is odd, then it is the sum of different powers of 2

If $k+1$ is even or $k+1$ is odd, $k+1$ is the sum of different powers of 2

Problem 4. [10 points] Let S be a set of ordered pair of integers defined recursively as follows.
1. $(0,0) \in S$
2. If $(a,b) \in S$, then $(a+1, b+3) \in S$ and $(a+3, b+1) \in S$
3. Nothing else is in S

a. [5 points] List the elements in S that result from applying the recursive rule 0, 1, 2, and 3 times

$(0,0)$
$(1,3), (3,1)$
$(2,6), (4,4), (6,2)$
$(3,9), (5,7), (7,5), (9,3)$

b. [5 points] Use structural induction to show that for all $(a,b) \in S$, $a+b$ is a multiple of 4.

1. Base case: $(0,0) \in S$

$(0,0) \in S$ and $0+0 = 4 \cdot 0$

2. Induction step: Prove that if $(a,b) \in S$ and $a+b$ is a multiple of 4, then $(a+1)+(b+3)$ is a multiple of 4 and $(a+3)+(b+1)$ is a multiple of 4

1. $(a,b) \in S$ and $a+b$ is a multiple of 4 Induction hypothesis
2. $a+b = 4i$ for some integer i
3. $a+b+4 = 4i+4$
4. $a+b+4 = 4(i+1)$
5. $(a+1)+(b+3) = 4(i+1)$ and $(a+3)+(b+1) = 4(i+1)$
6. $(a+1)+(b+3)$ is a multiple of 4 and $(a+3)+(b+1)$ is a multiple of 4

Problem 5. [15 points] Write down the first 6 elements of the following sequences where $n \in \{1, 2, 3 \dots\}$ and then give a recursive definition for a_n. For part c, express the first 6 elements as powers of 2.

a. [5 points] $a_n = 3n - 10$

-7, -4, -1, 2, 5, 8

$f(1) = -7$
$f(n+1) = 3 + f(n)$

b. [5 points] $a_n = (1 + (-1)^n)^n$

0, 4, 0, 16, 0, 64

$f(1) = 0$
$f(2) = 4$
$f(n+2) = 4f(n)$

c. [5 points] $a_n = 2^{n!}$

$2^1, 2^2, 2^6, 2^{24}, 2^{120}, 2^{720}$

$f(1) = 2$
$f(n+1) = f(n)^{n+1}$