Section 2.6
Proof By Contradiction

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- $p \equiv \left((\neg p) \rightarrow \mathrm{F}\right)$

p	$\neg p$	$(\neg p) \rightarrow \mathrm{F}$	$p \leftrightarrow \left((\neg p) \rightarrow \mathrm{F}\right)$
T	F	T	T
F	T	F	T

Proofs by Contradiction

- Example: There are an infinite number of positive integers
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  2. There is a finite number of positive integers
  3. There is a largest positive integer. Call it x.
  4. $x + 1$ is a positive integer and $x + 1 > x$
  5. Line 4 contradicts line 3
  6. Therefore, there is an infinite number of positive integers

Proofs by Contradiction

- Another example: If a and b are positive, then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$
- First note that in predicate logic, the theorem can be expressed as:

$$\forall a \forall b \left((a > 0 \land b > 0) \rightarrow \neg \left(\sqrt{a} + \sqrt{b} = \sqrt{a+b} \right) \right)$$

- Its negation is:

$$\exists a \exists b \left((a > 0 \land b > 0) \land \left(\sqrt{a} + \sqrt{b} = \sqrt{a+b} \right) \right)$$

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- Proof by contradiction:
  1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$
  2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a+b}^2$

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- Proof by contradiction:
  1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$
  2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a + b}^2$
  3. $a + 2\sqrt{ab} + b = a + b$

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- Proof by contradiction:
    1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$
    2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a+b}^2$
    3. $a + 2\sqrt{ab} + b = a + b$
    4. $2\sqrt{ab} = 0$

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  1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$
  2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a+b}^2$
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  4. $2\sqrt{ab} = 0$
  5. $ab = 0$

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- Another example: If a and b are positive, then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$
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  1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$
  2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a+b}^2$
  3. $a + 2\sqrt{ab} + b = a + b$
  4. $2\sqrt{ab} = 0$
  5. $ab = 0$
  6. $a = 0$ or $b = 0$

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- Proof by contradiction:
    1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$
    2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a+b}^2$
    3. $a + 2\sqrt{ab} + b = a + b$
    4. $2\sqrt{ab} = 0$
    5. $ab = 0$
    6. $a = 0$ or $b = 0$
    7. Line 6 contradicts line 1

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- Another example: If a and b are positive, then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$
- Proof by contradiction:
  1. Assume a and b are positive and $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$
  2. $\left(\sqrt{a} + \sqrt{b}\right)^2 = \sqrt{a+b}^2$
  3. $a + 2\sqrt{ab} + b = a + b$
  4. $2\sqrt{ab} = 0$
  5. $ab = 0$
  6. $a = 0$ or $b = 0$
  7. Line 6 contradicts line 1
  8. Therefore, If a and b are positive, then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proofs by Contradiction

- Still another example: Prove that if $3n + 2$ is odd, then n is odd
- Note that the negation of the theorem is:

$$3n + 2 \text{ is odd and } n \text{ is not odd}$$

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    1. Assume that it is not the case that if $3n + 2$ is odd, then n is odd

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  1. Assume that it is not the case that if $3n + 2$ is odd, then n is odd
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- Proof by contradiction
  1. Assume that it is not the case that if $3n + 2$ is odd, then n is odd
  2. $3n + 2$ is odd and n is not odd
  3. $3n + 2$ is odd and n is even

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    4. $3n + 2$ is odd and $n = 2k$ for some integer k

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  4. $3n + 2$ is odd and $n = 2k$ for some integer k
  5. $3(2k) + 2$ is odd

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  5. $3(2k) + 2$ is odd
  6. $6k + 2$ is odd

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  2. $3n + 2$ is odd and n is not odd
  3. $3n + 2$ is odd and n is even
  4. $3n + 2$ is odd and $n = 2k$ for some integer k
  5. $3(2k) + 2$ is odd
  6. $6k + 2$ is odd
  7. $2(3k + 1)$ is odd

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  3. $3n + 2$ is odd and n is even
  4. $3n + 2$ is odd and $n = 2k$ for some integer k
  5. $3(2k) + 2$ is odd
  6. $6k + 2$ is odd
  7. $2(3k + 1)$ is odd
  8. $2(3k + 1)$ is odd and $2(3k + 1)$ is even

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  2. $3n + 2$ is odd and n is not odd
  3. $3n + 2$ is odd and n is even
  4. $3n + 2$ is odd and $n = 2k$ for some integer k
  5. $3(2k) + 2$ is odd
  6. $6k + 2$ is odd
  7. $2(3k + 1)$ is odd
  8. $2(3k + 1)$ is odd and $2(3k + 1)$ is even
  9. Therefore, if $3n + 2$ is odd, then n is odd