Section 4.5
Composition of Functions

Function Composition

- Assume functions $f: A \to B$ and $g: B \to C$
- Create a new function $h: A \to C$
  - where $h(a) = g(f(a))$
  - Function h is the composition of functions f and g
  - Instead of writing $h(a) = g(f(a))$, we can write $h = g \circ f$
    - $(g \circ f)(a) = g(f(a))$

Function Composition

- The composition of functions can be described by a diagram
- Example
  - $A = \{a, b, c\}$
  - $B = \{1, 2, 3, 4\}$
  - $C = \{bear, cat, dog\}$

$f: A \to B$	$g: B \to C$
$f(a) = 3$	$g(1) = cat$
$f(b) = 1$	$g(2) = dog$
$f(c) = 4$	$g(3) = bear$
	$g(4) = bear$

Function Composition

- Example: Assume that f is a function that maps movie categories to popularity and g is a function that maps movie titles to movie categories

$$f\text{: Movie}-\text{Category} \rightarrow \{\text{popular}, \text{not}-\text{popular}\}$$

$$g\text{: Movie}-\text{Title} \rightarrow \text{Movie}-\text{Category}$$

$$f(\text{science}-\text{fiction}) = \text{popular}$$

$$g(\text{"Star Wars"}) = \text{science}-\text{fiction}$$

Function Composition

- Example continued: Then $f \circ g$ is a function that maps movie titles to their popularity

$$(f \circ g)(\text{"Star Wars"}) = f(g(\text{"Star Wars"})) = \text{popular}$$

Function Composition

- Another example: Assume $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ and $g: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ where

$$f(x) = 2x + 3$$

$$g(x) = 3x + 2$$

- Then

$$(f \circ g) \; = f\big(g(x)\big)$$

Function Composition

- Another example: Assume $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ and $g: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ where

$$f(x) = 2x + 3$$
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- Then

$$\begin{aligned}(f \circ g) &= f\big(g(x)\big) \\ &= 2\big(g(x)\big) + 3\end{aligned}$$

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$$\begin{aligned}(f \circ g) &= f\big(g(x)\big) \\ &= 2\big(g(x)\big) + 3 \\ &= 2(3x + 2) + 3 \\ &= (6x + 4) + 3\end{aligned}$$

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$$\begin{aligned}(f \circ g) &= f\big(g(x)\big) \\ &= 2\big(g(x)\big) + 3 \\ &= 2(3x + 2) + 3 \\ &= (6x + 4) + 3 \\ &= 6x + 7\end{aligned}$$

Function Composition

- Another example continued
- However

$$(g \circ f) \ = g\big(f(x)\big)$$

Function Composition

- Another example continued
- However

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\begin{aligned}
(g \circ f) &= g\big(f(x)\big) \\
&= 3\big(f(x)\big) + 2
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Function Composition

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- However

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\begin{aligned}
(g \circ f) &= g\big(f(x)\big) \\
&= 3\big(f(x)\big) + 2 \\
&= 3(2x + 3) + 2 \\
&= (6x + 9) + 2
\end{aligned}
$$

Function Composition

- Another example continued
- However

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\begin{aligned}
(g \circ f) &= g\big(f(x)\big) \\
&= 3\big(f(x)\big) + 2 \\
&= 3(2x + 3) + 2 \\
&= (6x + 9) + 2 \\
&= 6x + 11
\end{aligned}
$$

Inverse of Function Composition

- Assume $f: B \rightarrow C$ and $g: A \rightarrow B$ are one-to-one correspondences and thus are invertible

$$(f \circ g)^{-1}(y) = x \text{ where } (f \circ g)(x) = y$$

Inverse of Function Composition

- Assume $f: B \to C$ and $g: A \to B$ are one-to-one correspondences and thus are each invertible

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x &= g^{-1}\big(f^{-1}(y)\big) \\
x &= (g^{-1} \circ f^{-1})(y)
\end{aligned}
$$

- Thus $(f \circ g)^{-1}(y) = (g^{-1} \circ f^{-1})\ (y)$

Example 1

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If $f \circ g$ is one-to-one, must be f one-to-one? Must g be one-to-one?

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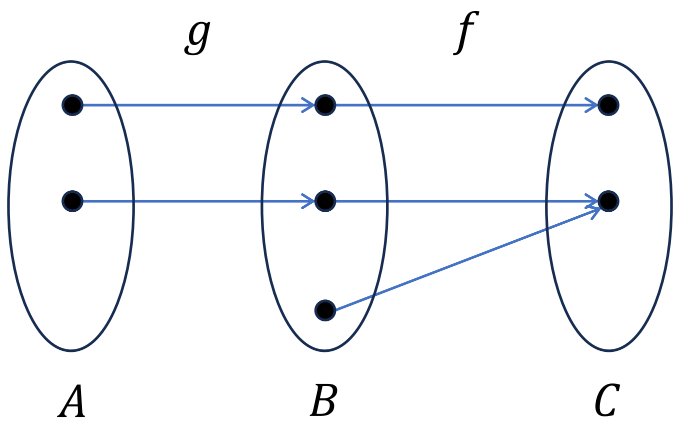
If $f \circ g$ is one-to-one, must be f one-to-one? Must g be one-to-one?

No, f does not have to be one-to-one



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1. Assume $f \circ g$ is one-to-one
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3. There are $a \in A$ and $b \in A$ such that $a \neq b$ and $g(a) = g(b)$

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4. $f(g(a)) = f(g(b))$

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4. $f(g(a)) = f(g(b))$
5. $f \circ g(a) = f \circ g(b)$
6. $f \circ g$ is not one-to-one which contradicts $f \circ g$ being one-to-one

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5. $f \circ g(a) = f \circ g(b)$
6. $f \circ g$ is not one-to-one which contradicts $f \circ g$ being one-to-one
7. g is one-to-one

Example 2

Assume $f: B \rightarrow C$ and $g: A \rightarrow B$

If $f \circ g$ is onto, must be f onto? Must g be onto?

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Yes, f must be onto. Proof by contradiction:

1. Assume $f \circ g$ is onto
2. Assume f is not onto
3. There is a $c \in C$ such that there is no $b \in B$ such that $f(b) = c$

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3. There is a $c \in C$ such that there is no $b \in B$ such that $f(b) = c$
4. Then there is no $a \in A$ such that $f(g(a)) = c$

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5. $f \circ g$ is not onto which contradicts the assumption that it is onto

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6. f is onto

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Assume $f: B \rightarrow C$ and $g: A \rightarrow B$

If $f \circ g$ is onto, must be f onto? Must g be onto?

No, g does not have to be onto.