

# Section 8.11

## Induction and Recursive Algorithms

# String Reversal

Name: ReverseString(s)

Input: string s

Output: the reverse of s

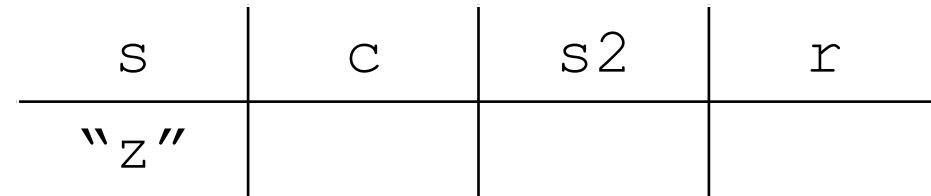
```
if (s = "")
    return ""
else
    c := first character of s
    s2 := the result of removing the first character from s
    r := ReverseString(s2)
    return rc, the result of adding c to the end of r
end-if
```

# String Reversal

Name: ReverseString(s)

Input: string s

Output: the reverse of s



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```
end-if
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s	c	s2	r
"z"	z		

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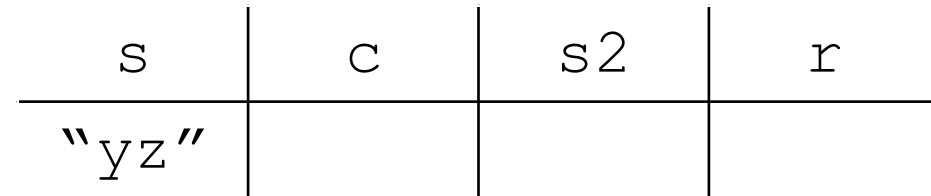
```
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```

```
end-if
```



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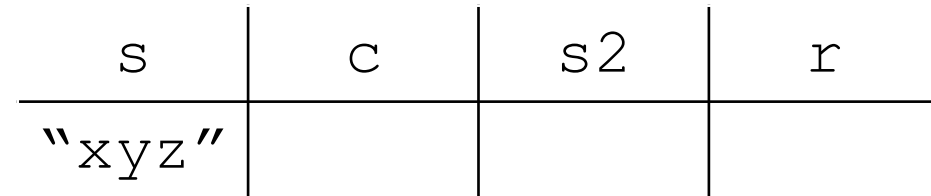
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end-if
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end-if
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end-if
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"xyz"	x	"yz"	"zy"

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    s2 := the result of removing the first character from s
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    r := ReverseString(s2)
```

```
    return rc, the result of adding c to the end of r
```

```
end-if
```

# The Correctness of ReverseString

- Prove by mathematical induction on the length of  $s$  that `ReverseString(s)` returns the reversal of  $s$
- Base case: The length of  $s$  is 0

By the `ReverseString` algorithm, `ReverseString("")` returns ""

# The Correctness of ReverseString

- Induction step: The length of  $s$  is  $k+1$



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  1. Assume that for all strings of length  $k$ , ReverseString returns the reversed string

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  3. ReverseString( $s$ ) returns  $c_{k+1}\dots c_2$  followed by  $c_1$  by the I.H.

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  3. ReverseString( $s$ ) returns  $c_{k+1}\dots c_2$  followed by  $c_1$  by the I.H.
  4. ReverseString( $s$ ) returns  $c_{k+1}\dots c_2c_1$

# Exponentiation

Name: Exponent(a, n)

Input: real number a and nonnegative integer n

Output:  $a^n$

```
if n = 0
    return 1
else
    return a * power(a, n-1)
end-if
```

# The Correctness of Exponent

- Prove by mathematical induction on the natural number  $n$  that  $\text{Exponent}(a, n)$  returns  $a^n$
- Base case:  $n$  is 0
  1.  $\text{Exponent}(a, 0)$  returns 1
  2.  $\text{Exponent}(a, 0)$  returns  $a^0$

# The Correctness of Exponent

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  3.  $\text{Exponent}(a, k+1)$  returns  $a * a^k$  by the I.H.
  4.  $\text{Exponent}(a, k+1)$  returns  $a^{k+1}$

# The Subsets of $\{a, b, c\}$ and $\{a, b, c\} \cup \{d\}$

The subsets of $\{a, b, c\}$	The subsets of $\{a, b, c\}$ combined with $d$
$\emptyset$	$\emptyset \cup \{d\}$
$\{a\}$	$\{a\} \cup \{d\}$
$\{b\}$	$\{b\} \cup \{d\}$
$\{c\}$	$\{c\} \cup \{d\}$
$\{a, b\}$	$\{a, b\} \cup \{d\}$
$\{a, c\}$	$\{a, c\} \cup \{d\}$
$\{b, c\}$	$\{b, c\} \cup \{d\}$
$\{a, b, c\}$	$\{a, b, c\} \cup \{d\}$
The subsets of $\{a, b, c\} \cup \{d\}$ that do not contain $d$	The subsets of $\{a, b, c\} \cup \{d\}$ that contain $d$

# Power Set

Name: PowerSet(s)  
Input: a finite set S  
Output: the power set of S

```
if S := ∅  
  return {∅}  
else  
  select an element a in S  
  S' := S - {a}  
  P' := PowerSet(S')  
  P := P'  
  for each x in P'  
    add x ∪ {a} to P  
  end-for  
  return P  
end-if
```

S	a	S'	P'	P	x
{1}					

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{1}	1				

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S	a	S'	P'	P	x
{1}	1	{}	{ $\emptyset$ }	{ $\emptyset$ , {1}}	$\emptyset$

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S	a	S'	P'	P	x
{1,2}					

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S	a	S'	P'	P	x
{1,2}	2				

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    add x ∪ {a} to P  
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end-if
```

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{1,2}	1	{2}			

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S	a	S'	P'	P	x
{1,2}	2	{1}	{ $\emptyset$ , {1}}		

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{1,2}	2	{1}	{ $\emptyset$ , {1}}	{ $\emptyset$ , {1}}	$\emptyset$

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{1,2}	2	{1}	{∅, {1}}	{∅, {1}, {2}}	∅

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S	a	S'	P'	P	x
{1,2}	2	{1}	{ $\emptyset$ , {1}}	{ $\emptyset$ , {1}}	$\emptyset$
				{ $\emptyset$ , {1}, {2}}	{1}

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  end-for
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end-if
  
```

S	a	S'	P'	P	x
{1,2}	2	{1}	{∅, {1}}	{∅, {1}}	∅
				{∅, {1}, {2}}	{1}
				{∅, {1}, {2}, {1,2}}	

# The Correctness of PowerSet

- Prove by mathematical induction on the natural number  $n$  that for any set  $S$  of  $n$  elements  $\text{PowerSet}(S)$  returns the power set of  $S$
- Base case:  $n$  is 0 and  $S = \emptyset$   
 $\text{PowerSet}(\emptyset)$  returns  $\{\emptyset\}$

# The Correctness of PowerSet

- Induction step
  1. Assume that PowerSet returns the power set of any set of  $k$  elements
  2. Let  $S$  be a set of  $k+1$  elements and let  $a$  be the element selected from  $S$  by the PowerSet algorithm
  3. First, show that  $\text{PowerSet}(S) \subseteq \mathcal{P}(S)$
  4. Assume  $T \in \text{PowerSet}(S)$
  5. There are 2 cases:  $a \notin T$  and  $a \in T$

# The Correctness of PowerSet

6. Case 1:  $a \notin T$
7.  $T \in P' = \text{PowerSet}(S') = \mathcal{P}(S')$  by the I.H.
8.  $T \subseteq S' \subseteq S$
9.  $T \in \mathcal{P}(S)$
10. Case 2:  $a \in T$
11.  $T$  was added to PowerSet( $S$ ) by adding  $a$  to an element of  $P'$
12. By the I.H.,  $P' = \mathcal{P}(S')$  so  $T \in \mathcal{P}(S)$  since  $a \in S$
13. In each case,  $T \in \mathcal{P}(S)$
14.  $\text{PowerSet}(S) \subseteq \mathcal{P}(S)$

# The Correctness of PowerSet

15. Second, show that  $\mathcal{P}(S) \subseteq \text{PowerSet}(S)$
16. Assume  $T \in \mathcal{P}(S)$
17. There are 2 cases:  $a \notin T$  and  $a \in T$
18. Case 1:  $a \notin T$
19.  $T \subseteq S'$  and hence  $T \in \mathcal{P}(S')$
20.  $T \in \text{PowerSet}(S')$  by the I.H.
21.  $T \in \text{PowerSet}(S)$  since  $S' \subseteq S$



# The Correctness of PowerSet

22. Case 2:  $a \in T$
23.  $T - \{a\} \subseteq S'$  and hence  $T - \{a\} \in \mathcal{P}(S')$
24.  $T - \{a\} \in \text{PowerSet}(S') = P'$  by the I.H.
25.  $(T - \{a\}) \cup \{a\} = T$  is added to  $P$  in the for loop
26.  $T \in \text{PowerSet}(S)$
27. In each case,  $T \in \text{PowerSet}(S)$
28.  $\mathcal{P}(S) \subseteq \text{PowerSet}(S)$
29.  $\text{PowerSet}(S) = \mathcal{P}(S)$