So far, we have considered permutations and combinations, we did not allow for the same object to be selected twice. How do things change if we allow objects to be selected multiple times?

Type	Repetitions Allowed	Formula
r-perm	No	$P(n,r) = \frac{n!}{(n-r)!}$
r-comb	No	$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$
r-perm	Yes	n^r
r-comb	Yes	

Combinations with Repetition: Three schools A, B, and C are competing for a grand prize in a science fair competition. There are two judges who anonymously recommends one of the three schools to win. Calculate the number of ways the judges can recommend the schools.

A		B		C
✓✓	|		|	
✓	|	✓	|	
✓	|		|	✓
	|	✓✓	|	
	|	✓	|	✓
	|		|	✓✓

Each row is an arrangement of 2 checks and 2 bars.

_ _ _ _

4 total positions

We count each of the ways of placing the two checkmarks in the 4 positions. Each one corresponds with one of the voting options.

$$\Rightarrow \binom{4}{2}$$

Theorem: $\binom{n-1+r}{r}$ is the number of r-combinations of n elements with repetition.

Intuitively, $n-1$ is the # of bars, and r is the number of checks.

Section 6.5 Problem 9: A bagel shop has 8 different kinds of bagels. How many different ways are there to choose:

- 6 bagels?

$n=8, r=6$ $\qquad \binom{8-1+6}{6} = \binom{13}{6}$

- 12 bagels?

$n=8, r=12$ $\qquad \binom{8-1+12}{12} = \binom{19}{12}$

- 12 bagels with at least one of each kind?

We must choose one of each type. This is 8 of our 12 bagels. Go ahead and choose one of each type. The remaining 4 can be chosen any way.

$n=8, r=4$ $\qquad \binom{8-1+4}{4} = \binom{11}{4}$

- 12 bagels with at least 3 egg bagels and no more than 2 salty bagels?

Eggs are handled similarly to last time; Choose the 3 egg bagels immediately and focus on the remaining 9 choices.

There are 3 options for the remaining 9:

Salty	non-salty	Options
0	9	$\binom{7-1+9}{9} = \binom{15}{9}$
1	8	$\binom{7-1+8}{8} = \binom{14}{8}$
2	7	$\binom{7-1+7}{7} = \binom{13}{7}$

Sum Rule: $\binom{15}{9} + \binom{14}{8} + \binom{13}{7}$

Permutations of indistinguishable objects: There may be cases where we want to permute objects in a set, but two different objects may appear to be the same. In such cases, there may be 2 different permutations of the objects which we want to count as the same permutation.

What is the number of strings obtained by permuting the letters of the word "RAD"?

$3 \times 2 \times 1 = 3! = 6$

Alternative Way

R - can be placed in $\binom{3}{1}$ positions.

A - can be placed in $\binom{2}{1}$ positions.

D - can be placed in $\binom{1}{1}$ position.

$$\binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{3!}{\cancel{2!} \cdot 1!} \cdot \frac{\cancel{2!}}{1!\,1!} \cdot \frac{1!}{0!\,1!} = 3!$$

What is the number of strings obtained by permuting the letters of the word “RADAR”?

R - 2 of them - $\binom{5}{2}$ ways of placing the Rs.

A - 2 of them - $\binom{3}{2}$ ways of placing the As.

D - 1 of them - $\binom{1}{1}$ ways of placing the D.

$$\binom{5}{2}\cdot\binom{3}{2}\cdot\binom{1}{1} = \frac{5!}{\cancel{3!}\,2!}\cdot\frac{\cancel{3!}}{2!\,1!}\cdot\frac{1!}{1!\,0!} = \frac{5!}{2!\cdot 2!}$$

Theorem 3: The number of permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k such that $n_1 + n_2 + \cdots + n_k = n$ is:

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}$$

Distributing objects into boxes: Many counting problems can be solved by enumerating the ways objects can be placed into boxes, where the order of placing objects within a box does not matter.

Objects
- Distinguishable (DO)
- Indistinguishable (IO)

Boxes
- Distinguishable (DB)
- Indistinguishable (IB)

Distinguishable objects into distinguishable boxes (DODB)

Example: count the number of 5 card poker hands for 4 players in a game. Assume a standard, 52-card deck is used.

player 1 : $\binom{52}{5}$ 47 cards left

player 2 : $\binom{47}{5}$ 42 left

player 3 : $\binom{42}{5}$ 37 left

player 4 : $\binom{37}{5}$ 32 left.

Total # of hands: $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}$

Theorem 4: The number of ways to distribute n distinguishable objects into k distinct boxes so that n_i objects are placed in box i, $i = 1, \dots, k$ and $n_1 + n_2 + \cdots + n_k = n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

Indistinguishable objects into distinguishable boxes (IODB)

Science fair judge promblem from earlier fits this description.

Distinguishable Boxes

- We can tell the difference between votes for A vs votes for B.

Indistinguishable Objects

- We cannot tell the difference between the judges.

Theorem 5: The number of ways to place r indistinguishable objects into n distinct boxes is:

$$\binom{n-1+r}{r}$$