CS 3333: Mathematical Foundations

Combinatorics

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- **The Product Rule**: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each way of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

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  - $5 \cdot 3 = 15$ ways.

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  - For each letter, there are 100 different numbers to choose from.

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- Example 2: Chairs are labeled with an uppercase letter followed by a number in the range 1 to 100. How many distinct ways are there to label a chair?
  - First, we choose the letter. There are 26 different ways to choose the letter.
  - For each letter, there are 100 different numbers to choose from.
  - There are $26 \cdot 100 = 2600$ different ways to label a chair.

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  - There are two options for each bit (0 or 1).
  - There are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$ ways.

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- Example 5: License plates have 3 letters followed by 4 digits. What is the total number of distinct license plates?

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  - There are 26 choices for each letter.
  - There are 10 choices for each digit.

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- Example 5: License plates have 3 letters followed by 4 digits. What is the total number of distinct license plates?
  - There are 26 choices for each letter.
  - There are 10 choices for each digit.
  - $26^3 \cdot 10^4$ different license plates.

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  - There are 26 options for the first letter.
  - There are 25 options for the second letter.

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  - There are 26 options for the first letter.
  - There are 25 options for the second letter.
  - There are 24 options for the third letter.

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- What if we are not allowed to use a letter more than once?
  - There are 26 options for the first letter.
  - There are 25 options for the second letter.
  - There are 24 options for the third letter.
  - Total: $26 \cdot 25 \cdot 24 \cdot 10^4$

Combinatorics

- Example 9: Calculate the value of k after executing the following pseudo code:

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-
```
k := 0
for i = 1 → 10 do
    for j = 1 → 20 do
        k++;
    end for
end for
```

Combinatorics

- Example 9: Calculate the value of k after executing the following pseudo code:
- $k := 0$
  for $i = 1 \to 10$ **do**
      for $j = 1 \to 20$ **do**
          $k++;$
      end for
  end for
- $k = 10 \cdot 20$

Combinatorics

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- $k := 0$
  for $i_1 = 1 \to n_1$ **do**
      for $i_2 = 1 \to n_2$ **do**
          $\dots$
          for $i_p = 1 \to n_p$ **do**
              $k++;$
          end for
      end for
  end for

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- $k := 0$
  for $i_1 = 1 \to n_1$ **do**
      for $i_2 = 1 \to n_2$ **do**
          $\dots$
          for $i_p = 1 \to n_p$ **do**
              $k++;$
          end for
      end for
  end for
- $k = n_1 \cdot n_2 \cdots n_p$

Combinatorics

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          k++;
      end for
  end for
  ```
  ```

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- $k := 0$
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- Notice that the inner for loop starts at i and not 1.

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- $k := 0$
  for $i = 1 \to 10$ **do**
      for $j = i \to 20$ **do**
          $k++;$
      end for
  end for
- Notice that the inner for loop starts at i and not 1.
- When $i = 1$, the inner loop iterates 20 times, when $i = 2$, the inner loop iterates 19 times, etc.

$20 + 19 + 18 + 17 + \ldots + 10$

Combinatorics

- **The Sum Rule**: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

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- Example: JPL cafeteria has 3 lunch choices and UC cafeteria has 10 lunch choices. How many different lunch choices are there?

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- Example: JPL cafeteria has 3 lunch choices and UC cafeteria has 10 lunch choices. How many different lunch choices are there?
  - $3 + 10 = 13$

Combinatorics

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- $k := 0$
  for $i_1 = 1 \to n_1$ **do**
  $k++$;
  end for
  for $i_2 = 1 \to n_2$ **do**
  $k++$;
  end for
  $\ldots$
  for $i_p = 1 \to n_p$ **do**
  $k++$;
  end for

Combinatorics

- Example 14: Calculate the value of k after executing the following pseudo code:
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k := 0
for i_1 = 1 → n_1 do
    k++;
end for
for i_2 = 1 → n_2 do
    k++;
end for
...
for i_p = 1 → n_p do
    k++;
end for
```
- $k = n_1 + n_2 + \cdots + n_p$

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- Example 16': Suppose a password for a computer account can be 6, 7, or 8 characters in length; the characters can be letters or digits. What is the number of ways a password can be formed?

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- Let p_6, p_7, and p_8 be the number of ways to form a 6, 7, or 8 character password respectively.
- What is p_6?

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- Let p_6, p_7, and p_8 be the number of ways to form a 6, 7, or 8 character password respectively.
- What is p_6?
    - There are 36 characters. 36^6 total.

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- Let p_6, p_7, and p_8 be the number of ways to form a 6, 7, or 8 character password respectively.
- What is p_6?
  - There are 36 characters. 36^6 total.
- Likewise, $p_7 = 36^7$ and $p_8 = 36^8$.

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- Let p_6, p_7, and p_8 be the number of ways to form a 6, 7, or 8 character password respectively.
- What is p_6?
  - There are 36 characters. 36^6 total.
- Likewise, $p_7 = 36^7$ and $p_8 = 36^8$.
- By the sum rule, the total number of passwords of length 6, 7, or 8 is $36^6 + 36^7 + 36^8$.

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What if we require there to be at least one digit in the password to be valid?

Total # of Passwords with no restrictions − Passwords we don't want = Passwords we want

$36^6 - 26^6 = 36^6 - 26^6$

$36^7 - 26^7 = 36^7 - 26^7$

$36^8 - 26^8 = 36^8 - 26^8$

Total: $36^6 + 36^7 + 36^8 - 26^6 - 26^7 - 26^8$

Combinatorics

- **Principle of Inclusion-Exclusion (PIE)**: If a task can be done in either n_1 ways or n_2 ways but the first group has n_3 things in common with the second group, then the number of ways to do the task is $n_1 + n_2 - n_3$.

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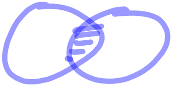

Example 18: How many bit strings of length 8 start with a 1 or end with 00?

1 x x x x x x x | x x x x x x 0 0

2^7 2^6

$2^7 + 2^6$ will overcount the following: 1 x x x x x 0 0

2^5

Final Count: $2^7 + 2^6 - 2^5$

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- **Division Rule**: If n ways are feasible to do a task, but each way is the same as d other ways, then there are n/d different ways to accomplish the task.

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- This rule can help us to ignore "unimportant" differences when counting things.

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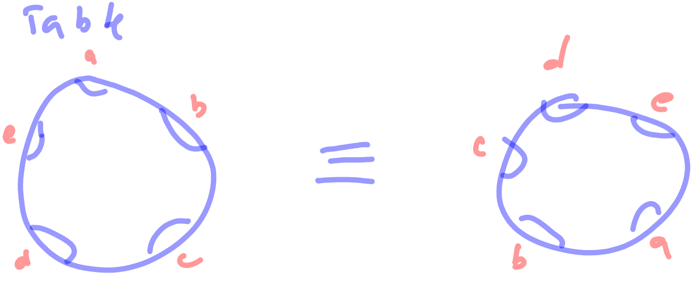
Example: 5 people standing in a line versus sitting around a table.

Standing in a line: $\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5!$



5! total options.

Each option is equivalent to 5 others.

Total $= \frac{5!}{5} = 4!$

Example: There are 10 different sandwiches and 3 different drinks. What is the number of ways to pick a lunch? If the choice of a drink does not matter, what is the number of ways to pick a lunch?