Logical Equivalence in Predicate Logic

- If two statements in predicate logic are logically equivalent, then they have the same truth value regardless of the meaning of their predicates.

- Example:

$$\forall x\big(P(x) \land Q(x)\big) \equiv \forall x P(x) \land \forall x Q(x)$$

Logical Equivalence in Predicate Logic

- Note that it is NOT the case that:

$$\forall x\big(P(x) \vee Q(x)\big) \equiv \forall x P(x) \vee \forall x Q(x)$$

- We can show that they are not logically equivalent by giving a domain of discourse and interpretations of P and Q such that the two statements have different truth values

Logical Equivalence in Predicate Logic

- Note that it is NOT the case that:

$$\forall x\big(P(x) \lor Q(x)\big) \equiv \forall x P(x) \lor \forall x Q(x)$$

- Let the domain of discourse be the integers, let $P(x)$ be true exactly when x is an even number, and let $Q(x)$ be true exactly when x is an odd number.
- With that interpretation, $\forall x\big(P(x) \lor Q(x)\big)$ is true
- But $\forall x P(x) \lor \forall x Q(x)$ is false

Negated Quantifiers

- "x is a parent"

$$\exists y P(x, y)$$

- Note that "x is not a parent“ is the same as saying, “For any person y, x is not the parent of y”:

$$\neg \exists y P(x, y) \equiv \forall y \neg P(x, y)$$

- In general, the negation of an existentially quantified statement is the same as universally quantifying the negation of the inner statement

Negated Quantifiers

- “Everybody loves z"

$$\forall x\ L(x, z)$$

- The negation of “Everybody loves z“ means that there is a person that does not love x:

$$\neg\forall x\ L(x, z) \equiv \exists x\ \neg L(x, z)$$

De Morgan's Laws for Quantified Statements

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- As with other laws $P(x)$ is a template that can be replaced with more complex statements. Example:

$$\neg\forall x\big(Q(x) \land R(x,y)\big) \equiv \exists x\neg\big(Q(x) \land R(x,y)\big)$$

De Morgan's Laws for Quantified Statements

- De Morgan's laws for conjunction and disjunction also can be used in predicate logic:

$$\neg\forall x\big(Q(x) \wedge R(x,y)\big) \equiv \exists x\, \neg\big(Q(x) \wedge R(x,y)\big)$$

$$\equiv \exists x\, \big(\neg Q(x) \vee \neg R(x,y)\big)$$