CS 3333: Mathematical Foundations

Number Theory

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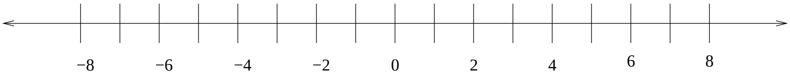
- The set of all integers is denoted by Z (i.e. $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$).



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- Sometimes is the set of all positive integers (i.e. $N = \{1, 2, 3, \ldots\}$).
- Sometimes is the set of all non-negative integers (i.e. $N = \{0, 1, 2, \ldots\}$).

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- If a is a positive integer, then $|a| = a$ (e.g. if $a = 13$, then $|a| = 13$).
- If a is a negative integer, then $|a| = -a$ (e.g. if $a = -13$, then $|a| = 13$).

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- If there is no integer c such that $b = ac$ then a does not divide b, denoted $a \nmid b$.

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- Example: Does 9 divide 36?
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- Example: Does 11 divide 120?
  - $11 \cdot 10 = 110$ and $11 \cdot 11 = 121$, so $11 \nmid 120$.

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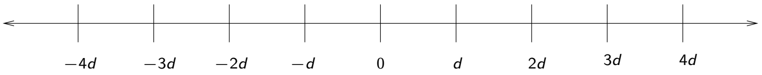

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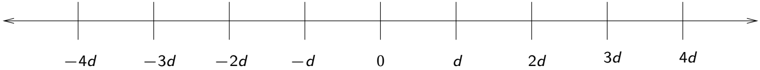

- We need to find an integer k such that
  1. $kd \leq n$
  2. $(k+1)d > n$

Number Theory

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- $\lceil x \rceil$ is the smallest integer $\geq x$ (ceiling function).
- $\lfloor x \rfloor$ is the largest integer $\leq x$ (floor function).
- Examples:
  $\lceil 11.7 \rceil = 12$; $\lfloor 11.7 \rfloor = 11$; $\lceil -5.3 \rceil = -5$; $\lfloor -5.3 \rfloor = -6$

Number Theory

- **Theorem 1**: Let a, b, and c be integers and $a \neq 0$.
  1. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.
  2. If $a \mid b$, then $a \mid bc$.
  3. If $a \mid b$ and $b \mid c$, $b \neq 0$, then $a \mid c$.
- How can we prove Theorem 1?

Number Theory

- **Problem 8 [KR] Sec. 4.1**: Prove or disprove: if $a \mid bc$, then $a \mid b$ or $a \mid c$.

Number Theory

- **Problem 8 [KR] Sec. 4.1**: Prove or disprove: if $a \mid bc$, then $a \mid b$ or $a \mid c$.
- **False.** Counterexample: $a = 4$, $b = 2$, and $c = 6$.

Number Theory

- **Problem 7 [KR] Sec. 4.1**: Let a, b, and c be integers such that $a \neq 0$ and $c \neq 0$. Prove or disprove: if $ac \mid bc$, then $a \mid b$.

Number Theory

- **Corollary 1**: Let a, b, and c be integers such that $a \neq 0$. If $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ whenever m and n are integers.

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- Terminology: $a =$ dividend, $d =$ divisor, $q =$ quotient, and $r =$ remainder.

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- $q = a \text{ div } d$ and $r = a \bmod d$.

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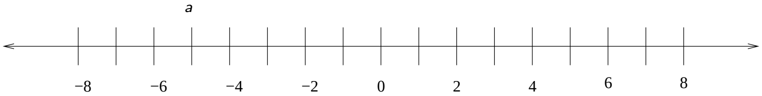
- The picture to have in mind:
- If $a < 0$:



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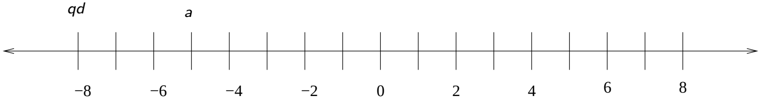

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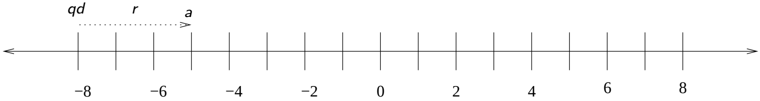

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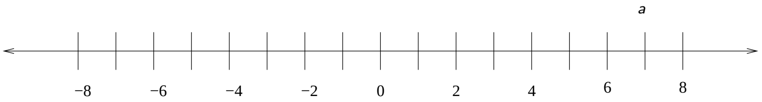

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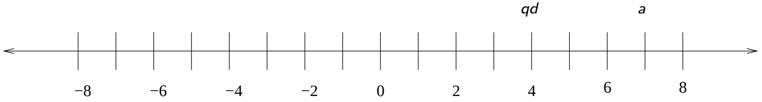

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