Section 7.2
Asymptotic Growth of Functions

Notation

- $\boldsymbol{Z}^+$ denotes the set of positive integers
  - $\boldsymbol{Z}^+ = \{1, 2, 3, \dots\}$

- $\boldsymbol{R}^{\geq}$ denotes the set of real numbers greater than or equal to 0
  - $\boldsymbol{R}^{\geq} = \{n \mid n \in \boldsymbol{R} \text{ and } n \geq 0\}$

Big-$\mathcal{O}$ Notation

- Let f and g be functions from the set $\boldsymbol{Z}^{+}$ to the set $\boldsymbol{R}^{\geq}$
  - $f: \boldsymbol{Z}^{+} \rightarrow \boldsymbol{R}^{\geq}$
  - g: $\boldsymbol{Z}^{+} \rightarrow \boldsymbol{R}^{\geq}$

- $f(n)$ is $\mathcal{O}\big(g(n)\big)$ if there are positive real constants c and n_0 such that

$$f(n) \leq c \cdot g(n)$$

  whenever $n \geq n_0$

  Such c and n_0 are called witnesses to the claim that $f(n)$ is $\mathcal{O}\big(g(n)\big)$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Example: Show that $7n^2$ is $\mathcal{O}\left(n^3\right)$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Example: Show that $7n^2$ is $\mathcal{O}(n^3)$

$$7n^2 \leq n \cdot n^2 \qquad \text{when } n \geq 7$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Example: Show that $7n^2$ is $\mathcal{O}(n^3)$

$$7n^2 \leq n \cdot n^2 \qquad \text{when } n \geq 7$$
$$7n^2 \leq 1 \cdot n^3 \qquad \text{when } n \geq 7$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Example: Show that $7n^2$ is $\mathcal{O}(n^3)$

$$7n^2 \leq n \cdot n^2 \qquad \text{when } n \geq 7$$

$$7n^2 \leq 1 \cdot n^3 \qquad \text{when } n \geq 7$$

- Thus $7n^2$ is $\mathcal{O}(n^3)$ with witnesses $c = 1$ and $n_0 = 7$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Another example: Show that $f(n) = n^2 + 2n + 1$ is $\mathcal{O}\left(n^2\right)$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Another example: Show that $f(n) = n^2 + 2n + 1$ is $\mathcal{O}(n^2)$
- Note that:

$$n^2 \leq n^2$$

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- Another example: Show that $f(n) = n^2 + 2n + 1$ is $\mathcal{O}(n^2)$
- Note that:

$$n^2 \leq n^2$$

$$2n \leq n^2 \quad \text{when } n \geq 2$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Another example: Show that $f(n) = n^2 + 2n + 1$ is $\mathcal{O}\left(n^2\right)$
- Note that:

$$n^2 \le n^2$$

$$2n \le n^2 \quad \text{when } n \ge 2$$

$$1 \le n^2 \quad \text{when } n \ge 1$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Another example: Show that $f(n) = n^2 + 2n + 1$ is $\mathcal{O}(n^2)$
- Note that:

$$
\begin{array}{rl}
n^2 \leq n^2 & \\
2n \leq n^2 & \text{when } n \geq 2 \\
1 \leq n^2 & \text{when } n \geq 1 \\
\hline
n^2 + 2n + 1 \leq 3n^2 & \text{when } n \geq \max(2, 1) = 2
\end{array}
$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Another example: Show that $f(n) = n^2 + 2n + 1$ is $\mathcal{O}\left(n^2\right)$
- Note that:

$$
\begin{array}{rl}
n^2 \leq n^2 & \\
2n \leq n^2 & \text{when } n \geq 2 \\
1 \leq n^2 & \text{when } n \geq 1 \\
\hline
n^2 + 2n + 1 \leq 3n^2 & \text{when } n \geq \max(2, 1) = 2
\end{array}
$$

$n^2 + 2n + 1$ is $\mathcal{O}\left(n^2\right)$ with witnesses $c = 3$ and $n_0 = 2$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $\mathcal{O}(n^2)$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $\mathcal{O}(n^2)$
- Note that:

$$3n^2 \leq 3n^2$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $\mathcal{O}(n^2)$
- Note that:

$$3n^2 \leq 3n^2$$

$$2n \leq 2n^2 \quad \text{when } n \geq 1$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $\mathcal{O}(n^2)$
- Note that:

$$3n^2 \leq 3n^2$$

$$2n \leq 2n^2 \quad \text{when} \quad n \geq 1$$

$$4 \leq 4n^2 \quad \text{when} \quad n \geq 1$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $\mathcal{O}(n^2)$
- Note that:

$$
\begin{array}{rl}
3n^2 \leq 3n^2 & \\
2n \leq 2n^2 & \text{when } n \geq 1 \\
4 \leq 4n^2 & \text{when } n \geq 1 \\
\hline
3n^2 + 2n + 4 \leq 9n^2 & \text{when } n \geq 1
\end{array}
$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Yet another example: Show that $f(n) = 3n^2 + 2n + 4$ is $\mathcal{O}(n^2)$
- Note that:

$$
\begin{array}{rl}
3n^2 \leq 3n^2 & \\
2n \leq 2n^2 & \text{when } \; n \geq 1 \\
4 \leq 4n^2 & \text{when } \; n \geq 1 \\
\hline
3n^2 + 2n + 4 \leq 9n^2 & \text{when } \; n \geq 1
\end{array}
$$

$3n^2 + 2n + 1$ is $\mathcal{O}(n^2)$ with witnesses $c = 9$ and $n_0 = 1$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Still another example: Show that $f(n) = 1 + 2 + \cdots n$ is $\mathcal{O}\left(n^2\right)$

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$$f(n) = 1 + 2 + \cdots n$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Still another example: Show that $f(n) = 1 + 2 + \cdots n$ is $\mathcal{O}(n^2)$

$$f(n) = 1 + 2 + \cdots n \leq \underbrace{n + n + \cdots + n}_{n \text{ times}}$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Still another example: Show that $f(n) = 1 + 2 + \cdots n$ is $\mathcal{O}(n^2)$

$$f(n) = 1 + 2 + \cdots n \leq \underbrace{n + n + \cdots + n}_{n \text{ times}} = n^2$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- Still another example: Show that $f(n) = 1 + 2 + \cdots n$ is $\mathcal{O}(n^2)$

$$f(n) = 1 + 2 + \cdots n \leq \underbrace{n + n + \cdots + n}_{n \text{ times}} = n^2$$

$f(n) = 1 + 2 + \cdots n$ is $\mathcal{O}(n^2)$ with witnesses $c = 1$ and $n_0 = 1$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- And another example: Show that $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$

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- And another example: Show that $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$

$$f(n) = 1 \cdot 2 \cdot \cdots \cdot n$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- And another example: Show that $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$

$$f(n) = 1 \cdot 2 \cdot \cdots \cdot n \leq \underbrace{n \cdot n \cdot \cdots \cdot n}_{n \text{ times}}$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- And another example: Show that $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$

$$f(n) = 1 \cdot 2 \cdot \cdots \cdot n \leq \underbrace{n \cdot n \cdot \cdots \cdot n}_{n \text{ times}} = n^n$$

Showing that $f(n)$ is $\mathcal{O}(g(n))$

- And another example: Show that $f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$

$$f(n) = 1 \cdot 2 \cdot \cdots \cdot n \leq \underbrace{n \cdot n \cdot \cdots \cdot n}_{n \text{ times}} = n^n$$

$f(n) = 1 \cdot 2 \cdot \cdots \cdot n$ is $\mathcal{O}(n^n)$ with witnesses $c = 1$ and $n_0 = 1$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- To show that $f(n)$ is not $\mathcal{O}(g(n))$, you must show that for any c and n_0, it is not the case that $f(n) \leq c \cdot g(n)$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction
  1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction
  1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0
  2. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction
  1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0
  2. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$
  3. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$, $n > 0$, and $n > c$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction
  1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0
  2. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$
  3. $n^2 \leq c \cdot n$ for all n when $n \geq n_0, n > 0$, and $n > c$
  4. $n \leq c$ for all n when $n \geq n_0, n > 0$, and $n > c$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction
  1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0
  2. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$
  3. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$, $n > 0$, and $n > c$
  4. $n \leq c$ for all n when $n \geq n_0$, $n > 0$, and $n > c$
  5. $n \leq c$ when $n > c$ is a contradiction

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Example: Show that n^2 is not $\mathcal{O}(n)$
- Proof: By contradiction
  1. Assume that n^2 is $\mathcal{O}(n)$ with witnesses c and n_0
  2. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$
  3. $n^2 \leq c \cdot n$ for all n when $n \geq n_0$, $n > 0$, and $n > c$
  4. $n \leq c$ for all n when $n \geq n_0$, $n > 0$, and $n > c$
  5. $n \leq c$ when $n > c$ is a contradiction
  6. n^2 is not $\mathcal{O}(n)$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction
  1. Assume that n^3 is $\mathcal{O}\left(7n^2\right)$ with witnesses c and n_0

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction
  1. Assume that n^3 is $\mathcal{O}\left(7n^2\right)$ with witnesses c and n_0
  2. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction
  1. Assume that n^3 is $\mathcal{O}\left(7n^2\right)$ with witnesses c and n_0
  2. $n^3 \le c \cdot 7n^2$ when $n \ge n_0$
  3. $n^3 \le c \cdot 7n^2$ when $n \ge n_0, n > 0$, and $n > c \cdot 7$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction
  1. Assume that n^3 is $\mathcal{O}(7n^2)$ with witnesses c and n_0
  2. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0$
  3. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0$, $n > 0$, and $n > c \cdot 7$
  4. $n \leq c \cdot 7$ when $n \geq n_0$, $n > 0$, and $n > c \cdot 7$

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction
  1. Assume that n^3 is $\mathcal{O}\left(7n^2\right)$ with witnesses c and n_0
  2. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0$
  3. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0$, $n > 0$, and $n > c \cdot 7$
  4. $n \leq c \cdot 7$ when $n \geq n_0$, $n > 0$, and $n > c \cdot 7$
  5. $n \leq c \cdot 7$ when $n > c \cdot 7$ is a contradiction

Showing that $f(n)$ is Not $\mathcal{O}(g(n))$

- Another example: Show that n^3 is not $\mathcal{O}(7n^2)$
- Proof: By contradiction
  1. Assume that n^3 is $\mathcal{O}\left(7n^2\right)$ with witnesses c and n_0
  2. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0$
  3. $n^3 \leq c \cdot 7n^2$ when $n \geq n_0, n > 0$, and $n > c \cdot 7$
  4. $n \leq c \cdot 7$ when $n \geq n_0, n > 0$, and $n > c \cdot 7$
  5. $n \leq c \cdot 7$ when $n > c \cdot 7$ is a contradiction
  6. n^3 is not $\mathcal{O}(7n^2)$

Big-Ω Notation

- Let f and g be functions from the set $\boldsymbol{Z}^+$ to the set $\boldsymbol{R}^{\geq}$
  - $f: \boldsymbol{Z}^+ \rightarrow \boldsymbol{R}^{\geq}$
  - g: $\boldsymbol{Z}^+ \rightarrow \boldsymbol{R}^{\geq}$

- $f(n)$ is $\Omega\left(g(n)\right)$ if there are <u>positive</u> real constants c and n_0 such that

$$f(n) \geq c \cdot g(n)$$

  whenever $n \geq n_0$

  Such c and n_0 are called witnesses to the claim that $f(n)$ is $\Omega\left(g(n)\right)$

Big-$\mathcal{O}$ and Big-Ω

If f and g be functions from the set $\boldsymbol{Z}^+$ to the set $\boldsymbol{R}^{\geq}$, then

$$f \text{ is } \mathcal{O}(g) \text{ if and only if } g \text{ is } \Omega(f)$$

Big-$\mathcal{O}$ and Big-Ω

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$$f \text{ is } \mathcal{O}(g) \text{ if and only if } g \text{ is } \Omega(f)$$

f is $\mathcal{O}(g)$ iff there are positive real c and n_0 such that $f(n) \leq c \cdot g(n)$ when $n \geq n_0$

Big-$\mathcal{O}$ and Big-Ω

If f and g be functions from the set $\boldsymbol{Z}^+$ to the set $\boldsymbol{R}^{\geq}$, then

$$f \text{ is } \mathcal{O}(g) \text{ if and only if } g \text{ is } \Omega(f)$$

f is $\mathcal{O}(g)$ iff there are positive real c and n_0 such that $f(n) \leq c \cdot g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $\frac{1}{c} \cdot f(n) \leq g(n)$ when $n \geq n_0$

Big-$\mathcal{O}$ and Big-$\boldsymbol{\Omega}$

If f and g be functions from the set $\boldsymbol{Z}^+$ to the set $\boldsymbol{R}^{\geq}$, then

f is $\mathcal{O}(g)$ if and only if g is $\Omega(f)$

f is $\mathcal{O}(g)$ iff there are positive real c and n_0 such that $f(n) \leq c \cdot g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $\frac{1}{c} \cdot f(n) \leq g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $g(n) \geq \frac{1}{c} \cdot f(n)$when $n \geq n_0$

Big-$\mathcal{O}$ and Big-Ω

If f and g be functions from the set $\boldsymbol{Z}^+$ to the set $\boldsymbol{R}^{\geq}$, then

$$f \text{ is } \mathcal{O}(g) \text{ if and only if } g \text{ is } \Omega(f)$$

f is $\mathcal{O}(g)$ iff there are positive real c and n_0 such that $f(n) \leq c \cdot g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $\frac{1}{c} \cdot f(n) \leq g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $g(n) \geq \frac{1}{c} \cdot f(n)$when $n \geq n_0$

iff there are positive real c and n_0 such that $g(n) \geq c \cdot f(n)$ when $n \geq n_0$

Big-$\mathcal{O}$ and Big-Ω

If f and g be functions from the set $\boldsymbol{Z}^+$ to the set $\boldsymbol{R}^{\geq}$, then

$$f \text{ is } \mathcal{O}(g) \text{ if and only if } g \text{ is } \Omega(f)$$

f is $\mathcal{O}(g)$ iff there are positive real c and n_0 such that $f(n) \leq c \cdot g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $\frac{1}{c} \cdot f(n) \leq g(n)$ when $n \geq n_0$

iff there are positive real $1/c$ and n_0 such that $g(n) \geq \frac{1}{c} \cdot f(n)$ when $n \geq n_0$

iff there are positive real c and n_0 such that $g(n) \geq c \cdot f(n)$ when $n \geq n_0$

iff g is $\Omega(f)$

Big-Ω Notation

- Example: $f(n) = 8n^3 + 5n^2 + 7$ is $\Omega(g)$ where $g(n) = n^3$
- Proof:

  $8n^3 + 5n^2 + 7 \geq n^3$ whenever $n > 0$

  So, $8n^3 + 5n^2 + 7 \geq n^3$ is $\Omega(g)$ with witnesses $c = 1$ and $n_0 = 0$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g(n))$ where $g(n) = 8n^2$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$n^2 \geq \frac{1}{8} 8n^2$ for all n

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

n^2	$\geq \frac{1}{8} 8n^2$	for all n
$\frac{1}{3} n^2$	$\geq \frac{1}{3 \cdot 8} 8n^2$	for all n

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$

$$\frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$

$$\frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$

$$\frac{1}{3} n^2 \geq 4n \quad \text{when } n \geq 12$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$
$$\frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$
$$\frac{1}{3} n^2 \geq 4n \quad \text{when } n \geq 12$$

$$n^2 \geq 3 \cdot 2 \quad \text{when } n \geq 6$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$\begin{aligned} n^2 &\geq \frac{1}{8}8n^2 && \text{for all } n \\ \frac{1}{3}n^2 &\geq \frac{1}{3\cdot 8}8n^2 && \text{for all } n \end{aligned}$$

$$\begin{aligned} n^2 &\geq 3\cdot 4n && \text{when } n \geq 12 \\ \frac{1}{3}n^2 &\geq 4n && \text{when } n \geq 12 \end{aligned}$$

$$\begin{aligned} n^2 &\geq 3\cdot 2 && \text{when } n \geq 6 \\ \frac{1}{3}n^2 &\geq 2 && \text{when } n \geq 6 \end{aligned}$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$
$$\frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$
$$\frac{1}{3} n^2 \geq 4n \quad \text{when } n \geq 12$$

$$n^2 \geq 3 \cdot 2 \quad \text{when } n \geq 6$$
$$\frac{1}{3} n^2 \geq 2 \quad \text{when } n \geq 6$$

$$\frac{1}{3} n^2 + \frac{1}{3} n^2 + \frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$

$$\frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$

$$\frac{1}{3} n^2 \geq 4n \quad \text{when } n \geq 12$$

$$n^2 \geq 3 \cdot 2 \quad \text{when } n \geq 6$$

$$\frac{1}{3} n^2 \geq 2 \quad \text{when } n \geq 6$$

$$\frac{1}{3} n^2 + \frac{1}{3} n^2 + \frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$

$$n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$
$$\frac{1}{3}n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$
$$\frac{1}{3}n^2 \geq 4n \quad \text{when } n \geq 12$$

$$n^2 \geq 3 \cdot 2 \quad \text{when } n \geq 6$$
$$\frac{1}{3}n^2 \geq 2 \quad \text{when } n \geq 6$$

$$\frac{1}{3}n^2 + \frac{1}{3}n^2 + \frac{1}{3}n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$
$$n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$
$$n^2 - 4n - 2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{when } n \geq 12$$

Big-Ω Notation

- Another example: $f(n) = n^2 - 4n - 2$ is $\Omega(g)$ where $g(n) = 8n^2$
- Proof:

$$n^2 \geq \frac{1}{8} 8n^2 \quad \text{for all } n$$
$$\frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{for all } n$$

$$n^2 \geq 3 \cdot 4n \quad \text{when } n \geq 12$$
$$\frac{1}{3} n^2 \geq 4n \quad \text{when } n \geq 12$$

$$n^2 \geq 3 \cdot 2 \quad \text{when } n \geq 6$$
$$\frac{1}{3} n^2 \geq 2 \quad \text{when } n \geq 6$$

$$\frac{1}{3} n^2 + \frac{1}{3} n^2 + \frac{1}{3} n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$
$$n^2 \geq \frac{1}{3 \cdot 8} 8n^2 + 4n + 2 \quad \text{when } n \geq 12 \text{ and when } n \geq 6$$
$$n^2 - 4n - 2 \geq \frac{1}{3 \cdot 8} 8n^2 \quad \text{when } n \geq 12$$

$n^2 - 4n - 2$ is $\Omega(8n^2)$ with witnesses c $= \frac{1}{24}$ and $n_0 = 12$

Big-Θ Notation

- Let f and g be functions from either the set of integers or the set of real numbers to the set of real numbers
- $f(n)$ is $\Theta\big(g(n)\big)$ if $f(n)$ is $\mathcal{O}\big(g(n)\big)$ and $f(n)$ is $\Omega\big(g(n)\big)$

Big-Θ Notation

Theorem: If $f(n)$ is $\Theta\big(g(n)\big)$ then $g(n)$ is $\Theta\big(f(n)\big)$

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1. Assume $f(n)$ is $\Theta\big(g(n)\big)$

Big-Θ Notation

Theorem: If $f(n)$ is $\Theta\big(g(n)\big)$ then $g(n)$ is $\Theta\big(f(n)\big)$

Proof:

1. Assume $f(n)$ is $\Theta\big(g(n)\big)$
2. $f(n)$ is $\mathcal{O}\big(g(n)\big)$ and $f(n)$ is $\Omega\big(g(n)\big)$

Big-Θ Notation

Theorem: If $f(n)$ is $\Theta\big(g(n)\big)$ then $g(n)$ is $\Theta\big(f(n)\big)$

Proof:

1. Assume $f(n)$ is $\Theta\big(g(n)\big)$
2. $f(n)$ is $\mathcal{O}\big(g(n)\big)$ and $f(n)$ is $\Omega\big(g(n)\big)$
3. $g(n)$ is $\Omega\big(f(n)\big)$ and $g(n)$ is $\mathcal{O}\big(g(n)\big)$ (proved last lecture)

Big-Θ Notation

Theorem: If $f(n)$ is $\Theta\big(g(n)\big)$ then $g(n)$ is $\Theta\big(f(n)\big)$

Proof:

1. Assume $f(n)$ is $\Theta\big(g(n)\big)$
2. $f(n)$ is $\mathcal{O}\big(g(n)\big)$ and $f(n)$ is $\Omega\big(g(n)\big)$
3. $g(n)$ is $\Omega\big(f(n)\big)$ and $g(n)$ is $\mathcal{O}\big(g(n)\big)$ (proved last lecture)
4. $g(n)$ is $\Theta\big(f(n)\big)$

Big-Θ Notation

- Example: Show that $f(n) = 1 + 2 + \cdots + n$ is $\Theta(n^2)$
- Proof: A previous example showed that $f(n) = 1 + 2 + \cdots + n$ is $\mathcal{O}(n^2)$. To show that $f(n)$ is $\Theta(n^2)$, we just need to show that $f(n)$ is $\Omega(n^2)$

Big-Θ Notation

$f(n) \quad = 1 + 2 + \cdots + (n-1) + n$

Big-Θ Notation

$$f(n) = 1 + 2 + \cdots + (n-1) + n$$

$$\begin{aligned} 2 \cdot f(n) = &\ \ 1 + 2 + \cdots + (n-1) + n \\ &+1 + 2 + \cdots + (n-1) + n \end{aligned}$$

Big-Θ Notation

$$f(n) = 1 + 2 + \cdots + (n-1) + n$$

$$\begin{aligned} 2 \cdot f(n) = &\ 1 + 2 + \cdots + (n-1) + n \\ &+1 + 2 + \cdots + (n-1) + n \end{aligned}$$

$$\begin{aligned} 2 \cdot f(n) = &\ 1 + \quad 2 \quad + \cdots + (n-1) + n \\ &+ n + (n-1) + \cdots + \quad 2 \quad + 1 \end{aligned}$$

Big-Θ Notation

$$
\begin{aligned}
f(n) &= 1 + 2 + \cdots + (n-1) + n \\
2 \cdot f(n) &= \quad 1 + 2 + \cdots + (n-1) + n \\
&\quad +1 + 2 + \cdots + (n-1) + n \\
2 \cdot f(n) &= \quad 1 + \quad 2 \quad + \cdots + (n-1) + n \\
&\quad + n + (n-1) + \cdots + \quad 2 \quad + 1 \\
2 \cdot f(n) &= n(n+1)
\end{aligned}
$$

Big-Θ Notation

$$f(n) = 1 + 2 + \cdots + (n-1) + n$$

$$2 \cdot f(n) = \begin{array}{l} 1 + 2 + \cdots + (n-1) + n \\ +1 + 2 + \cdots + (n-1) + n \end{array}$$

$$2 \cdot f(n) = \begin{array}{l} 1 + \quad 2 \quad + \cdots + (n-1) + n \\ + n + (n-1) + \cdots + \quad 2 \quad + 1 \end{array}$$

$$2 \cdot f(n) = n(n+1)$$

$$2 \cdot f(n) = n^2 + n$$

Big-Θ Notation

$$
\begin{aligned}
f(n) &= 1 + 2 + \cdots + (n-1) + n \\
2 \cdot f(n) &= \;\; 1 + 2 + \cdots + (n-1) + n \\
&\quad +1 + 2 + \cdots + (n-1) + n \\
2 \cdot f(n) &= \;\; 1 + \quad 2 \quad + \cdots + (n-1) + n \\
&\quad + n + (n-1) + \cdots + \quad 2 \quad + 1 \\
2 \cdot f(n) &= n(n+1) \\
2 \cdot f(n) &= n^2 + n \\
f(n) &= \frac{1}{2} n^2 + \frac{1}{2} n
\end{aligned}
$$

Big-Θ Notation

$$f(n) = 1 + 2 + \cdots + (n-1) + n$$

$$\begin{aligned} 2 \cdot f(n) = {} & 1 + 2 + \cdots + (n-1) + n \\ & +1 + 2 + \cdots + (n-1) + n \end{aligned}$$

$$\begin{aligned} 2 \cdot f(n) = {} & 1 + \quad 2 \quad + \cdots + (n-1) + n \\ & + n + (n-1) + \cdots + \quad 2 \quad + 1 \end{aligned}$$

$$2 \cdot f(n) = n(n+1)$$

$$2 \cdot f(n) = n^2 + n$$

$$f(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$f(n) \geq \frac{1}{2}n^2 \quad \text{when } n \geq 1$$

Big-Θ Notation

$$f(n) = 1 + 2 + \cdots + (n-1) + n$$

$$\begin{aligned} 2 \cdot f(n) = &\ \ 1 + 2 + \cdots + (n-1) + n \\ &+1 + 2 + \cdots + (n-1) + n \end{aligned}$$

$$\begin{aligned} 2 \cdot f(n) = &\ \ 1 + \quad 2 \quad + \cdots + (n-1) + n \\ &+ n + (n-1) + \cdots + \quad 2 \quad + 1 \end{aligned}$$

$$2 \cdot f(n) = n(n+1)$$

$$2 \cdot f(n) = n^2 + n$$

$$f(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$f(n) \geq \frac{1}{2}n^2 \quad \text{when } n \geq 1$$

$f(n)$ is $\Omega\left(n^2\right)$ with witnesses $c = \frac{1}{2}$ and $n_0 = 1$.
Since $f(n)$ is also $\mathcal{O}\left(n^2\right)$, $f(n)$ is $\Theta\left(n^2\right)$

Big-Θ Notation

- Another example: Show that $3n^2 + 8\mathrm{n} \cdot \log(n)$ is $\Theta(n^2)$
- Note that when $n > 0,\ \log(n) < n$

Big-Θ Notation

- Another example: Show that $3n^2 + 8\mathrm{n} \cdot \log(n)$ is $\Theta(n^2)$
- Note that when $n > 0,\ \log(n) < n$

$$\log(n) \ < n \qquad\qquad \text{when } n \geq 1$$

Big-Θ Notation

- Another example: Show that $3n^2 + 8\mathrm{n} \cdot \log(n)$ is $\Theta(n^2)$
- Note that when $n > 0,\ \log(n) < n$

$$\log(n) < n \qquad \text{when } n \geq 1$$

$$8\mathrm{n} \cdot \log(n) \leq 8n^2 \qquad \text{when } n \geq 1$$

Big-Θ Notation

- Another example: Show that $3n^2 + 8\text{n} \cdot \log(n)$ is $\Theta(n^2)$
- Note that when $n > 0,\ \log(n) < n$

$$
\begin{aligned}
\log(n) &< n && \text{when } n \geq 1 \\
8\text{n} \cdot \log(n) &\leq 8n^2 && \text{when } n \geq 1 \\
3n^2 + 8\text{n} \cdot \log(n) &\leq 11n^2 && \text{when } n \geq 1
\end{aligned}
$$

Big-Θ Notation

- Another example: Show that $3n^2 + 8\mathrm{n} \cdot \log(n)$ is $\Theta(n^2)$
- Note that when $n > 0,\ \log(n) < n$

$$\log(n) \le n \qquad \text{when } n \ge 1$$

$$8\mathrm{n} \cdot \log(n) \le 8n^2 \qquad \text{when } n \ge 1$$

$$3n^2 + 8\mathrm{n} \cdot \log(n) \le 11n^2 \qquad \text{when } n \ge 1$$

So, $3n^2 + 8\mathrm{n} \cdot \log(n)$ is $\mathcal{O}(n^2)$ with witnesses $c = 11$ and $n_0 = 1$

Big-Θ Notation

- Another example continued

  Now we need to show that $3n^2 + 8\mathrm{n} \cdot \log(n)$ is $\Omega(n^2)$

Big-Θ Notation

- Another example continued

  Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

$$\log(n) \geq 0 \qquad \text{when } n \geq 1$$

Big-Θ Notation

- Another example continued

  Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

$$\log(n) \geq 0 \qquad \text{when } n \geq 1$$

$$8n \cdot \log(n) \geq 0 \qquad \text{when } n \geq 1$$

Big-Θ Notation

- Another example continued

  Now we need to show that $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$

$$
\begin{aligned}
\log(n) &\geq 0 && \text{when } n \geq 1 \\
8n \cdot \log(n) &\geq 0 && \text{when } n \geq 1 \\
3n^2 + 8n \cdot \log(n) &\geq n^2 && \text{when } n \geq 1
\end{aligned}
$$

Big-Θ Notation

- Another example continued

  Now we need to show that $3n^2 + 8n \cdot \log(x)$ is $\Omega(n^2)$

$$
\begin{aligned}
\log(n) &\geq 0 && \text{when } n \geq 1 \\
8n \cdot \log(n) &\geq 0 && \text{when } n \geq 1 \\
3n^2 + 8n \cdot \log(n) &\geq n^2 && \text{when } n \geq 1
\end{aligned}
$$

  Hence $3n^2 + 8n \cdot \log(n)$ is $\Omega(n^2)$ with witnesses $c = 1$ and $n_0 = 1$

  Since $3n^2 + 8n \cdot \log(n)$ is also $\mathcal{O}(n^2)$, $3n^2 + 8n \cdot \log(n)$ is $\Theta(n^2)$

Asymptotic Growth of Polynomials

- Theorem 7.2.2

  Let $p(n)$ be a polynomial of degree k:

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n^1 + a_0$$

  where $a_k > 0$

  Then, $p(n)$ is $\Theta(n^k)$

Asymptotic Growth Logarithmic Functions of Different Bases

- If a and b constants such $a > 1$ and $b > 1$, then
$$\log_a(n) \text{ is } \Theta\big(\log_b(n)\big)$$
- Proof:
  1. $n = a^{log_a(n)} = b^{log_b(n)}$ when $n \geq 1$
  2. $\log_a\big(a^{log_a(n)}\big) = \log_a\big(b^{log_b(n)}\big)$ when $n \geq 1$
  3. $\log_a(n) = \log_a(b) \cdot \log_b(n)$ where $\log_a(b)$ is positive and $n \geq 1$
  4. $\log_a(n) = c \cdot \log_b(n)$ where $c = \log_a(b)$ is positive and $n \geq 1$
  5. $\log_a(n) \leq \mathrm{c} \cdot \log_b(n)$ where c is positive and $n \geq 1$
  6. $\log_a(n) \geq \mathrm{c} \cdot \log_b(n)$ where c is positive and $n \geq 1$
  7. $\log_a(n)$ is $\Theta\big(\log_b(n)\big)$

Growth Rates of Common Functions

- A function is a constant function if it always returns the same value
- If $f(n)$ is a constant function, then $f(n)$ is $\Theta(1)$
- $f(n)$ is called linear if $f(n)$ is $\Theta(n)$
- $f(n)$ is called polynomial if $f(n)$ is $\Theta(n^k)$ for a real number $k > 0$
- $f(n)$ is called exponential if $f(n)$ is $\Theta(c^n)$ for a real number $c > 1$

Common Functions in Algorithmic Complexity

Function	Name
$\Theta(1)$	Constant
$\Theta(\log(\log(n)))$	Log log
$\Theta(\log(n))$	Logarithmic
$\Theta(n)$	Linear
$\Theta(n \cdot \log(n))$	n log n
$\Theta(n^2)$	Quadratic
$\Theta(n^3)$	Cubic
$\Theta(n^k)\ k > 3$	Power
$\Theta(c^n)\ c > 1$	Exponential
$\Theta(n!)$	Factorial

Rules for Asymptotic Growth of Functions

- If $f(n)$ is $\mathcal{O}(h(n))$ and $g(n)$ is $\mathcal{O}(h(n))$, then $f(n) + g(n)$ is $\mathcal{O}(h(n))$
- If $f(n)$ is $\Omega(h(n))$ and $g(n)$ is $\Omega(h(n))$, then $f(n) + g(n)$ is $\Omega(h(n))$
- If $f(n)$ is $\mathcal{O}(g(n))$ and $a > 0$, then $a \cdot f(n)$ is $\mathcal{O}(g(n))$
- If $f(n)$ is $\Omega(g(n))$ and $a > 0$, then $a \cdot f(n)$ is $\Omega(g(n))$
- If $f(n)$ is $\mathcal{O}(g(n))$ and $g(n)$ is $\mathcal{O}(h(n))$, then $f(n)$ is $\mathcal{O}(h(n))$
- If $f(n)$ is $\Omega(g(n))$ and $g(n)$ is $\Omega(h(n))$, then $f(n)$ is $\Omega(h(n))$