Section 8.6
Strong Induction and Well-Ordering

Strong Induction

- Suppose we wish to prove $\forall n\ P(n)$ by mathematical induction
- In the induction step, to prove $P(k) \rightarrow P(k+1)$, we may need more than just the induction hypothesis $P(k)$ in order to derive $P(k+1)$
- Strong Induction:
  - Prove $\left(P(0) \land P(1) \land \cdots \land P(k)\right) \rightarrow P(k+1)$
    - The induction hypothesis is $P(0) \land P(1) \land \cdots \land P(k)$
    - Alternative induction hypothesis: $\forall i(i \leq k \rightarrow P(i))$

Proofs by Strong Induction

- Example: Prove $\forall n P(n)$ by strong induction where $P(n)$ is:

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Since $n > 1$, use strong induction on the set $\{2, 3, 4, \ldots\}$

Proofs by Strong Induction

1. Base case: Prove $P(2)$

   2 is prime and can written as the product of one prime, itself

Proofs by Strong Induction

2. Induction step: Prove $P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$

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2. Induction step: Prove $P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$

    1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
    2. $k+1$ is either prime or it is not prime

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    1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
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    3. Case 1: $k+1$ is prime

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    4. $k+1$ is the product of one prime number, itself

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   1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
   2. $k+1$ is either prime or it is not prime
   3. Case 1: $k+1$ is prime
   4. $k+1$ is the product of one prime number, itself
   5. Case 2: $k+1$ is not prime

Proofs by Strong Induction

2. Induction step: Prove $P(2) \wedge \cdots \wedge P(k) \to P(k+1)$

    1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
    2. $k+1$ is either prime or it is not prime
    3. Case 1: $k+1$ is prime
    4. $k+1$ is the product of one prime number, itself
    5. Case 2: $k+1$ is not prime
    6. $k+1 = a \cdot b$ where a and b are integers greater than 1 and less than $k+1$

Proofs by Strong Induction

2. Induction step: Prove $P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$

   1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
   2. $k+1$ is either prime or it is not prime
   3. Case 1: $k+1$ is prime
   4. $k+1$ is the product of one prime number, itself
   5. Case 2: $k+1$ is not prime
   6. $k+1 = a \cdot b$ where a and b are integers greater than 1 and less than $k+1$
   7. $2 \leq a \leq k$ and $2 \leq b \leq k$

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2. Induction step: Prove $P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$

    1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
    2. $k+1$ is either prime or it is not prime
    3. Case 1: $k+1$ is prime
    4. $k+1$ is the product of one prime number, itself
    5. Case 2: $k+1$ is not prime
    6. $k+1 = a \cdot b$ where a and b are integers greater than 1 and less than $k+1$
    7. $2 \leq a \leq k$ and $2 \leq b \leq k$
    8. a and b can be written as the product of one or more prime numbers (by the induction hypothesis)

Proofs by Strong Induction

2. Induction step: Prove $P(2) \land \cdots \land P(k) \rightarrow P(k+1)$

    1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
    2. $k+1$ is either prime or it is not prime
    3. Case 1: $k+1$ is prime
    4. $k+1$ is the product of one prime number, itself
    5. Case 2: $k+1$ is not prime
    6. $k+1 = a \cdot b$ where a and b are integers greater than 1 and less than $k+1$
    7. $2 \leq a \leq k$ and $2 \leq b \leq k$
    8. a and b can be written as the product of one or more prime numbers (by the induction hypothesis)
    9. $k+1 = a \cdot b$ can be written as the product of one or more prime numbers

Proofs by Strong Induction

2. Induction step: Prove $P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$

   1. Assume that for all i such that $2 \leq i \leq k$, i can be written as the product of one or more prime numbers
   2. $k+1$ is either prime or it is not prime
   3. Case 1: $k+1$ is prime
   4. $k+1$ is the product of one prime number, itself
   5. Case 2: $k+1$ is not prime
   6. $k+1 = a \cdot b$ where a and b are integers greater than 1 and less than $k+1$
   7. $2 \leq a \leq k$ and $2 \leq b \leq k$
   8. a and b can be written as the product of one or more prime numbers (by the induction hypothesis)
   9. $k+1 = a \cdot b$ can be written as the product of one or more prime numbers
   10. In both cases, $k+1$ can be written as the product of one or more primes

Strong Induction with Multiple Base Cases

- With strong induction, it may be necessary to use multiple base cases

Strong Induction with Multiple Base Cases

- Example: Prove $\forall n\ P(n)$ by strong induction where $P(n)$ is

Any amount of postage n that is 12 cents or more can be formed by using just 4-cent stamps and 5-cent stamps

Strong Induction with Multiple Base Cases

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Use strong induction on the set $\{12, 13, 14, \dots\}$

Strong Induction with Multiple Base Cases

1. Base cases: Prove $P(12)$, $P(13)$, $P(14)$, $P(15)$

   12 cents of postage is formed by three 4-cent stamps

   13 cents of postage is formed by two 4-cent stamps and one 5-cent stamp

   14 cents of postage is formed by one 4-cent stamp and two 5-cent stamps

   15 cents of postage is formed by three 5-cent stamps

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Strong Induction with Multiple Base Cases

- The set being used for induction is $\{12, 13, 14, \dots\}$
- The cases for the inductive step start at 16, so each can be expressed as $k + 4$ where $k \in \{12, 13, 14, \dots\}$
- The induction hypothesis is $P(12) \wedge P(13) \wedge \cdots \wedge P(k + 3)$

Strong Induction with Multiple Base Cases

2. Prove $P(12) \wedge P(13) \wedge \cdots \wedge P(k+3) \rightarrow P(k+4)$

Strong Induction with Multiple Base Cases

2. Prove $P(12) \wedge P(13) \wedge \cdots \wedge P(k+3) \rightarrow P(k+4)$
   1. Assume that when $12 \leq i \leq k+3$, i cents of postage can be formed from 4-cent stamps and 5-cent stamps.

Strong Induction with Multiple Base Cases

2. Prove $P(12) \wedge P(13) \wedge \cdots \wedge P(k+3) \rightarrow P(k+4)$
    1. Assume that when $12 \leq i \leq k+3$, i cents of postage can be formed from 4-cent stamps and 5-cent stamps.
    2. Thus, postage for k cents can be formed by 4-cent and 5-cent stamps by the induction hypothesis

Strong Induction with Multiple Base Cases

2. Prove $P(12) \wedge P(13) \wedge \cdots \wedge P(k+3) \rightarrow P(k+4)$
   1. Assume that when $12 \leq i \leq k+3$, i cents of postage can be formed from 4-cent stamps and 5-cent stamps.
   2. Thus, postage for k cents can be formed by 4-cent and 5-cent stamps by the induction hypothesis
   3. Postage for $k+4$ cents can be formed by adding a 4-cent stamp to the postage for k cents