CS 3333: Mathematical Foundations

Binomial Theorem

Consider all 4-bit strings

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0	1	0	0
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- Define $n_i = \#$ of bit strings with i 1s.
  - $n_0 = \binom{4}{0} = 1;$
  - $n_1 = \binom{4}{1} = 4;$
  - $n_2 = \binom{4}{2} = 6;$
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  - $n_4 = \binom{4}{4} = 1;$
- The total is
  - $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$
  - $= 1 + 4 + 6 + 4 + 1 = 16$

Binomial Expressions

- Binomial expression: a sum of two terms, e.g., $x + y$.
- Powers of binomial expressions: $(x + y)^4$, $(x + y)^3$, $(a - b)^{10}$.

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Binomial Theorem

- Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

  , where $n \geq 0$.
- $= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} y^n$
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- 12 times of x. $\binom{25}{12} x^{12} \cdot y^{13}$.
- The coefficient is $\binom{25}{12}$.
- $\binom{25}{12} = \binom{25}{13}$.

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- Assume $X = 2x$ and $Y = -3y$, then $(X + Y)^{25}$.
- The term is
- $\binom{25}{12}X^{12} \cdot Y^{13}$
- $= \binom{25}{12}(2x)^{12} \cdot (-3y)^{13}$
- $= \binom{25}{12}2^{12}(-3)^{13}x^{12} \cdot y^{13}$
- The coefficient is $-\binom{25}{12}2^{12} \cdot 3^{13}$

Binomial Theorem

- Corollary 1: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- Apply the binomial theorem on $(x+y)^n$ when $x = y = 1$.
- $(x+y)^n = (1+1)^n = 2^n$

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- $(x+y)^n = (1+1)^n = 2^n$
- $(1+1)^n$
- $= \binom{n}{0}1^n \cdot 1^0 + \binom{n}{1}1^{n-1} \cdot 1^1 + \binom{n}{2}1^{n-2} \cdot 1^2 + \ldots \binom{n}{n}1^0 \cdot 1^n$
- $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$
- Therefore, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Binomial Theorem

- Corollary 2: $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$
- Apply the binomial theorem on $(x+y)^n$ when $x = 1$, $y = -1$.
- $(x+y)^n = (1-1)^n = 0$

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- Apply the binomial theorem on $(x+y)^n$ when $x = 1$, $y = -1$.
- $(x+y)^n = (1-1)^n = 0$
- $(1-1)^n$
- $= \binom{n}{0}1^n \cdot (-1)^0 + \binom{n}{1}1^{n-1} \cdot (-1)^1 + \ldots + \binom{n}{n}1^0 \cdot (-1)^n$
- $= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$
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- $(1-1)^n$
- $= \binom{n}{0}1^n \cdot (-1)^0 + \binom{n}{1}1^{n-1} \cdot (-1)^1 + \ldots + \binom{n}{n}1^0 \cdot (-1)^n$
- $= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$
- Therefore, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.
- $n = 5$, $(1-1)^5 = \binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} = 1 - 5 + 10 - 10 + 5 - 1 = 0$
- $n = 6$, $(1-1)^6 = \binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6} = 1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$

Pascal's Identity

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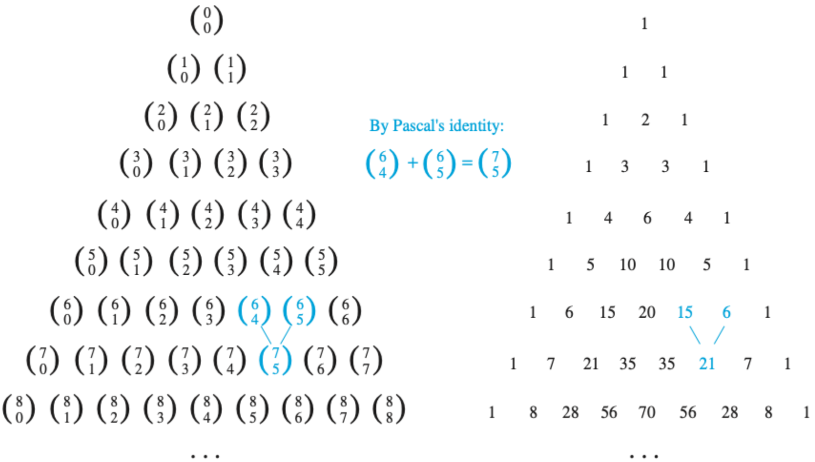
- $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
- Pascal's triangle



Proof of Pascal's Identity

- $\binom{n}{k-1} + \binom{n}{k}$
- $= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!}$
- $= \frac{n! \cdot k}{(n-k+1)!(k-1)! \cdot k} + \frac{n! \cdot (n-k+1)}{(n-k)! \cdot (n-k+1)k!}$
- $= \frac{n!(k+n-k+1)}{(n-k+1)!k!}$
- $= \frac{(n+1)!}{(n+1-k)!k!}$
- $= \binom{n+1}{k}$

Password Counting Problem

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- $36^6 - 26^6$
- If we consider the number of digits:
- 0 digit: $\binom{6}{0}10^0 \cdot 26^6$
- 1 digits: $\binom{6}{1}10^1 \cdot 26^5$
- 2 digits: $\binom{6}{2}10^2 \cdot 26^4$
- 3 digits: $\binom{6}{3}10^3 \cdot 26^3$
- 4 digits: $\binom{6}{4}10^4 \cdot 26^2$
- 5 digits: $\binom{6}{5}10^5 \cdot 26^1$
- 6 digits: $\binom{6}{6}10^6 \cdot 26^0$
- The total is $(10 + 26)^6$ by the binomial theorem.
- Then, we get $36^6 - 26^6$.

Theorem 3: Vandermonde's Identity (VI) [K.R. 6.4.3]

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

▸ Wikipedia

Combinatorial proof:

- $0 \leq r \leq min(m, n)$.

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- For example, we select a committee of r people from m men and n women.
- The direct answer is to select r people from $m + n$ people, $\binom{m+n}{r}$

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- $0 \leq r \leq min(m, n)$.
- For example, we select a committee of r people from m men and n women.
- The direct answer is to select r people from $m + n$ people, $\binom{m+n}{r}$
- Or, we select $k (\leq m)$ people from m men, and select $r - k$ people from n women, where $k \in [0, r]$.

Theorem 3: Vandermonde's Identity (VI)

- $0 \leq r \leq min(m, n)$.
- For example, $m = 5$, $n = 7$, and $r = 4$.
- $\binom{5+7}{4} = 495$
- $\sum_{k=0}^{4} \binom{5}{k}\binom{7}{4-k} =$ $\binom{5}{0}\binom{7}{4-0} + \binom{5}{1}\binom{7}{4-1} + \binom{5}{2}\binom{7}{4-2} + \binom{5}{3}\binom{7}{4-3} + \binom{5}{4}\binom{7}{4-4}$
- $= 35 + 5 * 35 + 10 * 21 + 10 * 7 + 5$
- $= 495$

Exercise

Show that if $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$.

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- Consider choosing a committee of k people with one designated chairman from n people.
- Method 1: Choose k committee members from n people, $\binom{n}{k}$. Then, choose one from k to be the chairman, $\binom{k}{1}$.

$$\binom{n}{k}\binom{k}{1} = k\binom{n}{k}$$

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- Method 2: First, choose the chairman from n people. Then choose the remains $k-1$ from $n-1$ people.

$$\binom{n}{1}\binom{n-1}{k} = n\binom{n-1}{k-1}$$

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- Formal proof:
- $LHS = k\binom{n}{k} = k\frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!}$
- $RHS = n\binom{n-1}{k-1} = n\frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$
- Therefore, $LHS = RHS$.
- Proved it.