Random Experiment (RE): an experiment whose outcome is not known in advance, but the set of all possible outcomes is known.

A **Sample Point** is an outcome of a RE.

The **Sample Space** is the set of all possible outcomes of a RE.

$\Omega = \{$ all possible outcomes of a RE $\}$

Flip of a coin

$\Omega = \{H, T\}$

Roll of a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Flip a coin until we get heads:

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

Example: Roll of a die.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Let A = event that the die roll results in a prime number.

$A = \{2, 3, 5\}$

B = event we roll a 6. $B = \{6\}$.

C = event that we roll a 7. $C = \{\} = \emptyset$

D = event that an even # occurs $D = \{2, 4, 6\}$

$A \cap D$ = event roll is prime and even $= \{2\}$

$\overline{A}$ = event that a prime does not occur $= \{1, 4, 6\}$

Probability Measure: assigns a number in [0,1] for the probability or chance that an event occurs.

A probability of 0 means it never occurs,

A probability of 1 means it always occurs.

All others between 0 and 1.

Classical Probability: Ω is finite, and each of the outcomes is equally likely to occur.

$$P(A) = \text{prob. event } A \text{ occurs} = \frac{|A|}{|\Omega|}$$

Example: Roll of a fair die.

Ex: Roll of a fair die. $|\Omega| = 6$

A = a prime occurs. $|A| = 3$

$$P(A) = \frac{3}{6} = \frac{1}{2}.$$

Example: roll a 7. $|C| = 0$

$$P(C) = \frac{|C|}{|\Omega|} = \frac{0}{6} = 0$$

Formal definition of probability measure:

Let F = family of events of Ω.

Probability measure P is a real-valued function on F.

$$P : F \to [0,1]$$

Axioms of probability measure:

1. $P(A) \geq 0$ for every event A.

2. $P(\Omega) = 1$

3. $P(A \cup B) = P(A) + P(B)$ if A & B are mutually exclusive
i.e, if $A \cap B = \emptyset$.

Example: A dresser contains 4 brown socks and 5 black socks. What is the probability that a brown sock is chosen if a sock is taken out of the drawer at random?

A = event that a brown sock is chosen.

$|\Omega| = 9 \qquad |A| = 4$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{9}$$

Example: Lottery, pick 4 digits from 0 to 9. Repetition is allowed, and order matters.

A - Grand Prize: if all 4 digits are chosen correctly.

B - Small Prize: if 3 of 4 digits are chosen correctly.

$|\Omega| = 10^4$

$|A| = 1 \implies P(A) = \frac{|A|}{|\Omega|} = \frac{1}{10^4}$

$|B| = \binom{4}{3} \cdot \binom{9}{1} = 36 \qquad P(B) = \frac{|B|}{|\Omega|} = \frac{36}{10^4}$

ways to choose 3 correct digits

ways to choose the last, incorrect digit.

Example: A box contains 50 identical objects labeled 1 to 50. What is the probabilty that the objects labeled 11, 4, 17, 39, and 23 are drawn in five consecutive random draws if

1. objects are replaced (put back into the box) after each draw? The 5 objects should be drawn in the order.

$|\Omega| = 50^5$

$|A| = 1$ $\qquad$ $P(A) = \frac{1}{50^5}$

2. objects are not replaced after they are drawn?

$|\Omega| = P(50,5) = \frac{50!}{45!}$

$|A| = 1$ $\qquad$ $P(A) = \frac{1}{P(50,5)} = \frac{45!}{50!}$

Theorems:

1. $P(\emptyset) = 0$
2. $P(\bar{A}) = 1 - P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: Find the probability that a 5-card poker hand contains four of a kind.

$$|\Omega| = \binom{52}{5}$$

$|A|$? Lets break it down.

4 of a kind in Aces : $\binom{48}{1}$

" " " " 2's : $\binom{48}{1}$

⋮

$$|A| = \binom{13}{1} \cdot \binom{48}{1} = 13 \cdot 48.$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{13 \cdot 48}{\binom{52}{5}}$$

Example: Suppose a sequence of 10 bits are randomly generated.

1. What is the probability that at least one of the bits is a 0?

$|\Omega| = 2^{10}$

A = event that at least one bit is 0.

$|A|$? Easier to count $\bar{A}$ = event that no bits are 0.

$|\bar{A}| = 1$. $P(A) = 1 - P(\bar{A}) = 1 - \frac{|\bar{A}|}{|\Omega|} = 1 - \frac{1}{2^{10}}$

2. What is the probability that exactly one of the bits is a 0?

$|\Omega| = 2^{10}$

$|A| = \binom{10}{1} = 10$

$P(A) = \frac{10}{2^{10}}$

Example: What is the probability that a randomly selected positive integer ≤ 100 is divisible by 2 or 5?

$|\Omega| = 100$

E_1 = event that sample point is divis by 2.

E_2 = event " " " " " 5.

We want $P(E_1 \cup E_2)$.

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$|E_1| = 50$. $|E_2| = 20$. $|E_1 \cap E_2| = 10$

$$P(E_1 \cup E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{60}{100} = .6$$