Section 8.15
Solving Linear Homogeneous Recurrence Relations

Solving a Recurrence Relation

- Recall that a recurrence relation describes a sequence:

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

$$f_0 = 0,\ f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

- A recurrence relation is solved if there is an explicit (or closed) formula that generates its terms without reference to previous terms

- A closed formula is a formula that uses a fixed number of terms

Solving a Recurrence Relation

- Example: $s_n = \sum_{i=0}^{n} i$ can be described as a recurrence relation:

$$s_0 = 0$$

$$s_n = s_{n-1} + n \text{ when } n \geq 1$$

This recurrence relation is solved by the following closed formula:

$$s_n = \frac{n(n+1)}{2}$$

Solving a Recurrence Relation

- Another example: Let g_n be defined by the following recurrence relation:

$$g_0 = 2$$

$$g_n = 5g_{n-1} \text{ when } n \geq 1$$

This recurrence relation is solved by the following closed formula:

$$g_n = 2 \cdot 5^n$$

This can be shown by induction on the natural numbers

Linear Homogeneous Recurrence Relations

- A linear homogeneous recurrence relation of degree k has the following form:

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}$$

Where:

- Each c_i is a constant
- $c_k \neq 0$

Linear because each c_i is a constant and each f_i is not raised to a power

Homogeneous because each term of the sum has the same form: $c_i f_{n-i}$

Linear Homogeneous Recurrence Relations

- The following are examples of linear homogenous recurrence relations

  - $P_n = (1.11)P_{n-1}$ (of degree 1)
  - $f_n = f_{n-1} + f_{n-2}$ (of degree 2)
  - $a_n = a_{n-5}$ (of degree 5)

Linear Homogeneous Recurrence Relations

- The following are NOT examples of linear homogenous recurrence relations

  - $a_n = a_{n-1} + a_{n-2}^2$
  - $H_n = 2H_{n-1} + 2$
  - $B_n = nB_{n-1}$

Linear Homogeneous Recurrence Relations

- The linear homogenous recurrence relation of degree k:

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}$$

has k initial conditions:

$$f_0 = C_0,\ \ f_1 = C_1, \ldots \quad f_{k-1} = C_{k-1}$$

Importance of Initial Conditions

- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, …

$$f_n = f_{n-1} + f_{n-2}$$

$$f_0 = 0, \quad f_1 = 1$$

- Lucas sequence: 2, 1, 3, 4, 7, 11, 18, …

$$l_n = l_{n-1} + l_{n-2}$$

$$l_0 = 2, \quad l_1 = 1$$

Linear Combinations of Sequences

- Both the Fibonacci and Lucas sequences satisfy the equation:

$$a_n = a_{n-1} + a_{n-2}$$

- If the values of the Fibonacci sequence are doubled, they still satisfy the equation:

$$f_n = f_{n-1} + f_{n-2}$$

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$$g_n = 2f_n$$

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- If the values of the Fibonacci sequence are doubled, they still satisfy the equation:

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ g_n &= 2f_n \\ &= 2(f_{n-1} + f_{n-2}) \end{aligned}$$

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Linear Combinations of Sequences

- If both the Fibonacci and Lucas sequences are multiplied by different constants and added together, their sum also satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad l_n = l_{n-1} + l_{n-2}$$

$$g_n = sf_n + tl_n$$

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$$\begin{aligned} g_n &= sf_n + tl_n \\ &= sf_{n-1} + sf_{n-2} + tl_{n-1} + tl_{n-2} \\ &= sf_{n-1} + tl_{n-1} + sf_{n-2} + tl_{n-2} \\ &= (sf_{n-1} + tl_{n-1}) + (sf_{n-2} + tl_{n-2}) \end{aligned}$$

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$$f_n = f_{n-1} + f_{n-2} \qquad\qquad l_n = l_{n-1} + l_{n-2}$$

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Linear Combinations of Sequences

- In general, if two sequences satisfy a linear homogeneous recurrence relation, then any linear combination of them also satisfies that linear homogeneous recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad\qquad l_n = l_{n-1} + l_{n-2}$$

$$\begin{aligned} g_n &= sf_n + tl_n \\ &= sf_{n-1} + sf_{n-2} + tl_{n-1} + tl_{n-2} \\ &= sf_{n-1} + tl_{n-1} + sf_{n-2} + tl_{n-2} \\ &= (sf_{n-1} + tl_{n-1}) + (sf_{n-2} + tl_{n-2}) \\ &= g_{n-1} + g_{n-2} \end{aligned}$$

Solving Linear Homogeneous Recurrence Relations

- From the earlier example:

$$g_0 = 2$$

$$g_n = 5g_{n-1} \text{ when } n \geq 1$$

We see that $g_n = 5g_{n-1}$ suggests a solution of the form: $g_n = 5^n$ and that from the initial condition $g_0 = 2$, we conclude $g_n = 2 \cdot 5^n$

- We then guess that all explicit solutions of linear homogenous recurrence relations involve $a_n = r^n$ for some real number r

Solving Linear Homogeneous Recurrence Relations

- The guessed relationship: $a_n = r^n$ implies for a recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$:

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$$a_n \quad = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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$$\begin{aligned} a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \\ r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \\ \frac{1}{r^{n-k}} r^n &= \frac{1}{r^{n-k}} \left(c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \right) \end{aligned}$$

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a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \\
r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \\
\frac{1}{r^{n-k}} r^n &= \frac{1}{r^{n-k}} \left(c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \right) \\
r^k &= c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k
\end{aligned}
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a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \\
r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \\
\frac{1}{r^{n-k}} r^n &= \frac{1}{r^{n-k}} \left(c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \right) \\
r^k &= c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k \\
r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k &= 0
\end{aligned}$$

Solving Linear Homogeneous Recurrence Relations

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r is a root of the polynomial $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k$

Characteristic Equations and Characteristic Roots

- $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ is called the characteristic equation

- The solutions to $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ are called the characteristic roots

Solving Linear Homogeneous Recurrence Relations

- Example: What is the characteristic equation of $a_n = a_{n-1} + 2a_{n-2}$?

  Assuming $a_n = r^n$:

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$$r^n = r^{n-1} + 2r^{n-2}$$

$$\frac{1}{r^{n-2}}(r^n) = \frac{1}{r^{n-2}}\left(r^{n-1} + 2r^{n-2}\right)$$

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r^2 &= r^1 + 2r^0 \\
r^2 &= r + 2 \\
r^2 - r - 2 &= 0
\end{aligned}
$$

Solving Linear Homogeneous Recurrence Relations

- Another example: What is the characteristic equation of $a_n = 3a_{n-1} - 7a_{n-2}$?

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Completing the Solution

- Each characteristic root yields a value for r in the term r^n.
- We can then create a linear combination of the terms and use the initial conditions to find leading coefficients of the r^n terms

Example

- What is the solution to the following recurrence relation:

$$a_0 = 2$$
$$a_1 = 3$$
$$a_n = a_{n-1} + 2a_{n-2}$$

The characteristic equation is:

$$r^n = r^{n-1} + 2r^{n-2}$$
$$r^2 = r + 2$$
$$r^2 - r - 2 = 0$$

Example

- The characteristic equation can be factored and solved

$$\begin{aligned} r^2 - r - 2 &= 0 \\ (r-2)(r+1) &= 0 \\ r &= -1, 2 \end{aligned}$$

- Quadratic formula for $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

$$r = -1, 2$$

  - There are two solutions:

$$a_n = (-1)^n \qquad a_n = 2^n$$

  - Linear combinations of the two solutions are also solutions

Example

- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

$$r = -1, 2$$

$$a_n = s \cdot (-1)^n + t \cdot (2)^n$$

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Use the initial cases to solve for s and t

$$a_0 = 2$$
$$a_1 = 3$$

Example

- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

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$$a_n = s \cdot (-1)^n + t \cdot (2)^n$$

$$a_0 = 2 = s \cdot (-1)^0 + t \cdot (2)^0$$

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- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

$$r = -1, 2$$

$$a_n = s \cdot (-1)^n + t \cdot (2)^n$$

$$a_0 = 2 = s \cdot (-1)^0 + t \cdot (2)^0$$
$$= s + t$$

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$$r = -1, 2$$

$$a_n = s \cdot (-1)^n + t \cdot (2)^n$$

$$\begin{aligned} a_0 = 2 &= s \cdot (-1)^0 + t \cdot (2)^0 \\ &= s + t \\ a_1 = 3 &= s \cdot (-1)^1 + t \cdot (2)^1 \end{aligned}$$

Example

- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

$$
\begin{aligned}
r &= -1, 2 \\
\\
a_n &= s \cdot r^n + t \cdot r^n \\
a_n &= s \cdot (-1)^n + \ t \cdot (2)^n \\
\\
a_0 = 2 &= s \cdot (-1)^0 + \ t \cdot (2)^0 \\
&= s + \ t \\
a_1 = 3 &= s \cdot (-1)^1 + \ t \cdot (2)^1 \\
&= -s + 2\,t
\end{aligned}
$$

Example

- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

$$2 = s + t$$
$$3 = -s + 2t$$

Example

- Use the initial conditions to solve for the coefficients of the linear combination of r^n:

$$
\begin{aligned}
2 &= s + t \\
3 &= -s + 2t
\end{aligned}
$$

$$
\begin{aligned}
s &= 1/3 \\
t &= 5/3
\end{aligned}
$$

Example

- Substitute the values for s and t into the equation for a_n

$$a_n = 1/3 \cdot (-1)^n + \; 5/3 \cdot (2)^n$$

Example

- Check the solution

n	$a_0 = 2$ $a_1 = 3$ $a_n = a_{n-1} + 2a_{n-2}$	$a_n = \frac{1}{3} \cdot (-1)^n + \frac{5}{3} \cdot (2)^n$
0	2	$\frac{1}{3} \cdot (-1)^0 + \frac{5}{3} \cdot (2)^0 = 2$
1	3	$\frac{1}{3} \cdot (-1)^1 + \frac{5}{3} \cdot (2)^1 = 3$
2	$3 + 2 \cdot 2 = 7$	$\frac{1}{3} \cdot (-1)^2 + \frac{5}{3} \cdot (2)^2 = 7$
3	$7 + 2 \cdot 3 = 13$	$\frac{1}{3} \cdot (-1)^3 + \frac{5}{3} \cdot (2)^3 = 13$
4	$13 + 2 \cdot 7 = 27$	$\frac{1}{3} \cdot (-1)^4 + \frac{5}{3} \cdot (2)^4 = 27$
5	$27 + 2 \cdot 13 = 53$	$\frac{1}{3} \cdot (-1)^5 + \frac{5}{3} \cdot (2)^5 = 53$

Example Summary

1. Start with a recurrence relation with initial conditions:

$$a_0 = 2$$

$$a_1 = 3$$

$$a_n = a_{n-1} + 2a_{n-2}$$

2. Assume a solution starting from:

$$a_n = r^n$$

3. Derive the characteristic equation from the recurrence relation:

$$r^2 - r - 2 = 0$$

4. Solve the equation:

$$r = -1, 2$$

There will be as many roots as the degree of the recurrence relation

Example Summary

5. Express the solution as a linear combination of the original assumption $a_n = r^n$:

$$a_n = s \cdot (-1)^n + t \cdot (2)^n$$

6. Apply the initial conditions to get simultaneous equations

$$a_0 = 2 = s \cdot (-1)^0 + t \cdot (2)^0$$

$$a_1 = 3 = s \cdot (-1)^1 + t \cdot (2)^1$$

7. Solve the simultaneous equations to get the coefficients s and t

$$s = 1/3 \quad t = 5/3$$

8. Substitute to get the final solution

$$a_n = 1/3 \cdot (-1)^n + 5/3 \cdot (2)^n$$

Solving When a Root is Repeated

- If a root, r, appears twice as a solution to a polynomial, then both r^n and nr^n are solutions to the recurrence relation
- For each additional occurrence of a root include an additional factor of n: $r^n, nr^n, n^2r^n, n^3r^n$ …
- Example: What is the solution to the recurrence relation:

$$f_0 = 2$$

$$f_1 = 3$$

$$f_n = 4f_{n-1} - 4f_{n-2}$$

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- If a root, r, appears twice as a solution to a polynomial, then both r^n and nr^n are solutions to the recurrence relation
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- Example: What is the solution to the recurrence relation:

$$
\begin{aligned}
r^n &= 4r^{n-1} - 4r^{n-2} \\
r^n - 4r^{n-1} + 4r^{n-2} &= 0 \\
r^2 - 4r^1 + 4 &= 0 \\
(r-2)(r-2) &= 0 \\
r &= 2, 2
\end{aligned}
$$

Solving When a Root is Repeated

- If a root, r, appears twice as a solution to a polynomial, then both r^n and nr^n are solutions to the recurrence relation
- For each additional occurrence of a root include an additional factor of n: $r^n, nr^n, n^2r^n, n^3r^n$ …
- Example: What is the solution to the recurrence relation:

$$f_n = s(2)^n + tn(2)^n$$

$$\begin{aligned} f_0 = 2 &= s(2)^0 + t(0)(2)^0 \\ 2 &= s \\ f_1 = 3 &= s(2)^1 + t(1)(2)^1 \\ 3 &= 4 + 2t \\ -1/2 &= t \end{aligned}$$

$$f_n = 2 \cdot 2^n - (1/2)n2^n$$

Solving When a Root is Repeated

- If a root, r, appears twice as a solution to a polynomial, then both r^n and nr^n are solutions to the recurrence relation
- Another example: What is the solution to the recurrence relation with the following characteristic equation:

$$\begin{aligned}(r-2)^3\,(r-3)^2 &= 0\\ r &= 2, 2, 2, 3, 3\end{aligned}$$

$$a_n = s2^n + tn2^n + un^2 2^n + v3^n + wn3^n$$

Use the initial conditions to solve for s, t, u, v, w