# Section 8.3
# Summations

# Summations

- We can add the terms of a sequence together

$$a_m + a_{m+1} + \cdots + a_n$$

- We do not have to start at the beginning of a sequence
- We start by summing only a finite number of terms from the sequence

# Summations

- The sum $a_m + a_{m+1} + \cdots + a_n$ can be abbreviated as

$$\sum_{j=m}^{n} a_j$$

$$\textstyle\sum_{j=m}^{n} a_j$$

$$\textstyle\sum_{m \le j \le n} a_j$$

- $j$ is the index of summation, $m$ is the lower limit, $n$ is the upper limit

# Summations

- Example 1: Using $\Sigma$, express the sum

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}$$

$$\sum_{i=1}^{100}\frac{1}{i}$$

# Summations

- Example 2: Evaluate $\sum_{j=1}^{5} j^2$

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$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25\end{aligned}$$

# Summations

- Example 2: Evaluate $\sum_{j=1}^{5} j^2$

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55\end{aligned}$$

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$$\sum_{k=4}^{8}(-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$

# Summations

- Example 3: Evaluate $\sum_{k=4}^{8}(-1)^k$

$$
\begin{aligned}
\sum_{k=4}^{8}(-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\
&= 1 + (-1) + 1 + (-1) + 1
\end{aligned}
$$

# Summations

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$$\begin{aligned}\sum_{k=4}^{8}(-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1\end{aligned}$$

# Summations

- Example 4: Evaluate $\sum_{k=1}^{3} 5$

$$\sum_{k=1}^{3} 5$$

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$$\sum_{k=1}^{3} 5 = 5 + 5 + 5$$
$$= 15$$

# Shifting Indices in Summations

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- Example: Instead of going from 1 to 5, change a summation so that the index goes from 0 to 4

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$$= (0+1)^2 + (1+1)^2 + (2+1)^2 + (3+1)^2 + (4+1)^2$$

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- Sometimes, we need to shift the indices of a summation
- Example: Instead of going from 1 to 5, change a summation so that the index goes from 0 to 4

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$$= (0+1)^2 + (1+1)^2 + (2+1)^2 + (3+1)^2 + (4+1)^2$$

$$= \sum_{k=0}^{4} (k+1)^2$$

# Separating Final Terms in Summations

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$$\sum_{i=1}^{6} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6)$$
$$= (3(1) + 3(2) + 3(3) + 3(4) + 3(5)) + 3(6)$$

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- A summation can be simplified by separating the last term from the summation

$$\begin{aligned}\sum_{i=1}^{6} 3i &= 3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6) \\ &= (3(1) + 3(2) + 3(3) + 3(4) + 3(5)) + 3(6) \\ &= \left(\sum_{i=1}^{5} 3i\right) + 3(6)\end{aligned}$$

# Summation of Sums

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$$\sum_{i=1}^{n} f(i) + g(i) \;\; = (f(1) + g(1)) + (f(2) + g(2)) + \cdots + (f(n) + g(n))$$

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$$= (f(1) + f(2) + \cdots f(n)) + (g(1) + g(2) + \cdots + g(n))$$

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$$\sum_{i=1}^{n} f(i) + g(i) = (f(1) + g(1)) + (f(2) + g(2)) + \cdots + (f(n) + g(n))$$

$$= (f(1) + f(2) + \cdots f(n)) + (g(1) + g(2) + \cdots + g(n))$$

$$= \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$$

# Summation of Sums

Example:

$$\sum_{i=1}^{n} i^2 + 2i = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 2i$$

# Summations With Constant Factors

$$\sum_{i=1}^{n} c \cdot f(i)$$

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# Summations With Constant Factors

$$\sum_{i=1}^{n} c \cdot f(i) \quad = c \cdot f(1) + c \cdot f(2) + \cdots + c \cdot f(n) +$$

$$= c \cdot (f(1) + f(2) + \cdots + f(n))$$

$$= c \cdot \sum_{i=1}^{n} f(i)$$

# Summations With Constant Factors

Example:

$$\sum_{i=1}^{n} 5i^2 = 5\sum_{i=1}^{n} i^2$$

# Double Summations

- Terms in a summation can use more than one index
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$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} (i \cdot 1 + i \cdot 2 + i \cdot 3)$$

# Double Summations

- Terms in a summation can use more than one index
- Example:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i \cdot 1 + i \cdot 2 + i \cdot 3)$$

$$= (1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3)+(2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3)+(3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3)+(4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3)$$

# Double Summations

- Terms in a summation can use more than one index
- Example:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij \;=\; \sum_{i=1}^{4}(i \cdot 1 + i \cdot 2 + i \cdot 3)$$

$$= (1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3) + (2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3) + (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3) + (4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3)$$

$$= (1 + 2 + 3) + (2 + 4 + 6) + (3 + 6 + 9) + (4 + 8 + 12)$$

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- Example:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i \cdot 1 + i \cdot 2 + i \cdot 3)$$

$$= (1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3) + (2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3) + (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3) + (4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3)$$

$$= (1 + 2 + 3) + (2 + 4 + 6) + (3 + 6 + 9) + (4 + 8 + 12)$$

$$= 6 + 12 + 18 + 24$$

# Double Summations

- Terms in a summation can use more than one index
- Example:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i\cdot 1 + i\cdot 2 + i\cdot 3)$$

$$= (1\cdot 1 + 1\cdot 2 + 1\cdot 3)+(2\cdot 1 + 2\cdot 2 + 2\cdot 3)+(3\cdot 1 + 3\cdot 2 + 3\cdot 3)+(4\cdot 1 + 4\cdot 2 + 4\cdot 3)$$

$$= (1 + 2 + 3)+(2 + 4 + 6)+(3 + 6 + 9)+(4 + 8 + 12)$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

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- Same example: 2nd approach

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# Double Summations

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- Same example: 2nd approach

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i \cdot 1 + i \cdot 2 + i \cdot 3)$$

$$= \sum_{i=1}^{4} 6i$$

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- Same example: 2nd approach

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i \cdot 1 + i \cdot 2 + i \cdot 3)$$

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$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i \cdot 1 + i \cdot 2 + i \cdot 3)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

# Summations Over Sets

- The $\Sigma$ notation can also be used when computing sums involving members of a set
- Example: Let $f$ be a function from a set $S$ to the real numbers $f : S \to \boldsymbol{R}$

$$\sum_{s \in S} f(s)$$

means for each element $s \in S$, calculate $f(s)$ and then sum the results

# Summations Over Sets

- Example

$$\sum_{s\in\{2,3,5,7\}} 3s = (3 \cdot 2) + (3 \cdot 3) + (3 \cdot 5) + (3 \cdot 7)$$

$$= 6 + 9 + 15 + 21$$

$$= 51$$

# Useful Summation Formulas

**TABLE 2 Some Useful Summation Formulae.**

| *Sum* | *Closed Form* |
| --- | --- |
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1} - a}{r - 1}, r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$ | $\frac{1}{(1-x)^2}$ |

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} i$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i$$

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- Example

$$\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i$$

$$= \frac{100(101)}{2} - \frac{49(50)}{2}$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i$$

$$= \frac{100(101)}{2} - \frac{49(50)}{2}$$

$$= 5050 - 1225$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i$$

$$= \frac{100(101)}{2} - \frac{49(50)}{2}$$

$$= 5050 - 1225$$

$$= 3825$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} 3i - 5$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} 3i - 5 \quad = \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right)$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} 3i - 5 \quad = \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right)$$

$$= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right)$$

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- Example

$$\sum_{i=50}^{100} 3i - 5 = \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right)$$

$$= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right)$$

$$= \left(3\sum_{i=1}^{100} i - 100(5)\right) - \left(3\sum_{i=1}^{49} i - 49(5)\right)$$

# Useful Summation Formulas

- Example

$$
\begin{aligned}
\sum_{i=50}^{100} 3i - 5 &= \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right) \\
&= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right) \\
&= \left(3\sum_{i=1}^{100} i - 100(5)\right) - \left(3\sum_{i=1}^{49} i - 49(5)\right) \\
&= \left(3\,\frac{100(101)}{2} - 500\right) - \left(3\,\frac{49(50)}{2} - 245\right)
\end{aligned}
$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} 3i - 5 = \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right)$$

$$= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right)$$

$$= \left(3\sum_{i=1}^{100} i - 100(5)\right) - \left(3\sum_{i=1}^{49} i - 49(5)\right)$$

$$= \left(3\,\frac{100(101)}{2} - 500\right) - \left(3\,\frac{49(50)}{2} - 245\right)$$

$$= (3(5050) - 500) - (3(1225) - 245)$$

# Useful Summation Formulas

- Example

$$\sum_{i=50}^{100} 3i - 5 = \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right)$$

$$= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right)$$

$$= \left(3\sum_{i=1}^{100} i - 100(5)\right) - \left(3\sum_{i=1}^{49} i - 49(5)\right)$$

$$= \left(3\,\frac{100(101)}{2} - 500\right) - \left(3\,\frac{49(50)}{2} - 245\right)$$

$$= (3(5050) - 500) - (3(1225) - 245)$$

$$= (15150 - 500) - (3675 - 245)$$

# Useful Summation Formulas

- Example

$$
\begin{aligned}
\sum_{i=50}^{100} 3i - 5 \quad &= \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right) \\
&= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right) \\
&= \left(3\sum_{i=1}^{100} i - 100(5)\right) - \left(3\sum_{i=1}^{49} i - 49(5)\right) \\
&= \left(3\,\frac{100(101)}{2} - 500\right) - \left(3\,\frac{49(50)}{2} - 245\right) \\
&= (3(5050) - 500) - (3(1225) - 245) \\
&= (15150 - 500) - (3675 - 245) \\
&= 14650 - 3430
\end{aligned}
$$

# Useful Summation Formulas

- Example

$$
\begin{aligned}
\sum_{i=50}^{100} 3i - 5 &= \left(\sum_{i=1}^{100} 3i - 5\right) - \left(\sum_{i=1}^{49} 3i - 5\right) \\
&= \left(\sum_{i=1}^{100} 3i - \sum_{i=1}^{100} 5\right) - \left(\sum_{i=1}^{49} 3i - \sum_{i=1}^{49} 5\right) \\
&= \left(3\sum_{i=1}^{100} i - 100(5)\right) - \left(3\sum_{i=1}^{49} i - 49(5)\right) \\
&= \left(3\,\frac{100(101)}{2} - 500\right) - \left(3\,\frac{49(50)}{2} - 245\right) \\
&= (3(5050) - 500) - (3(1225) - 245) \\
&= (15150 - 500) - (3675 - 245) \\
&= 14650 - 3430 \\
&= 11220
\end{aligned}
$$