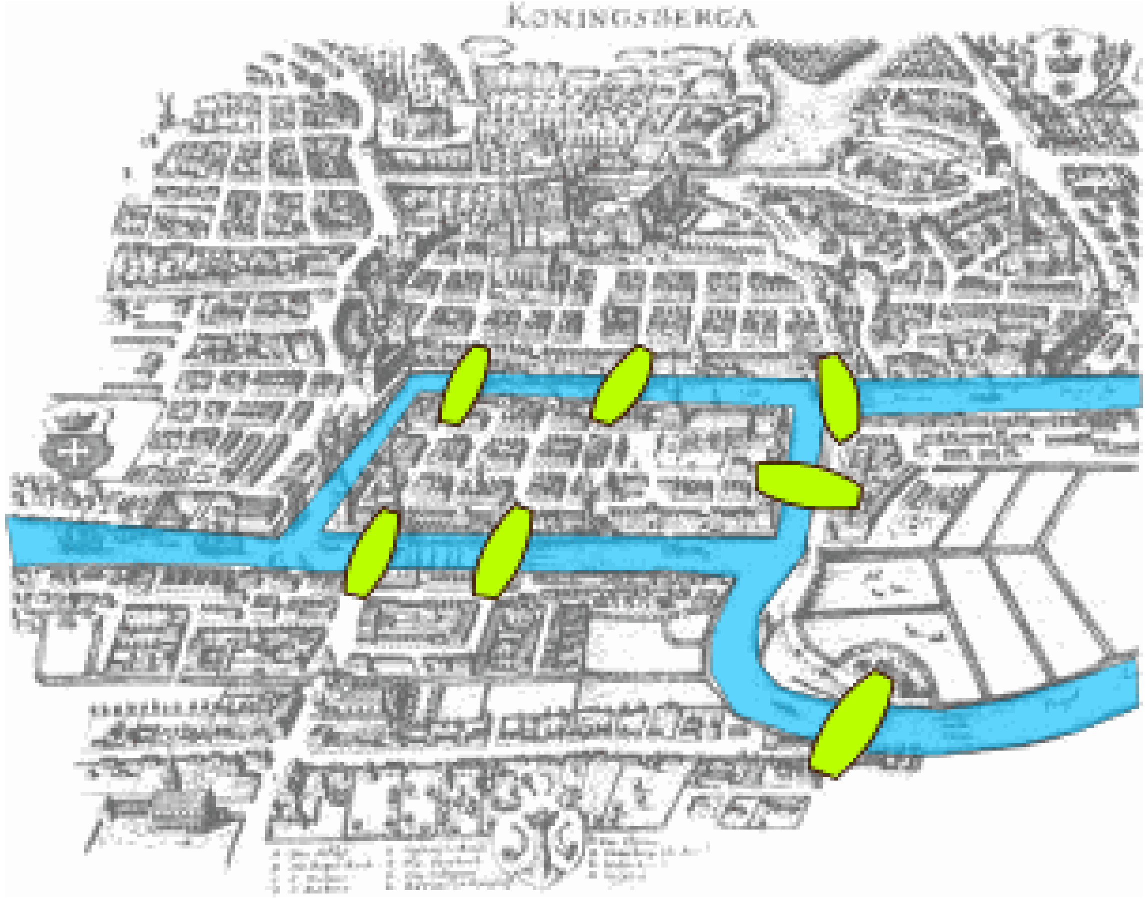
Section 13.1
Graphs

The Königsberg Bridge Problem

- The city of Königsberg had seven bridges
- Is there a path through the city that crosses each bridge exactly once?



Undirected Graphs

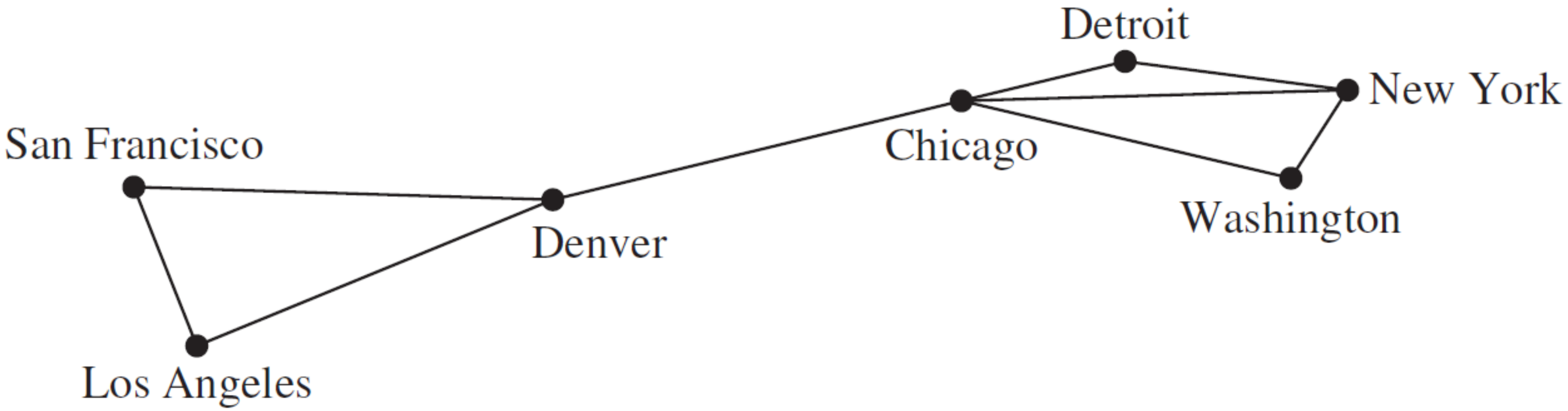
- An undirected graph $G = (V, E)$ consists of a non-empty set of vertices (or nodes), V, and a set of edges, E
  - Each edge in E is an unordered pair of vertices in V
  - Since edges are unordered pairs, edges do not have a direction
  - Each edge can be described as a two-element set. The edge $\{u, v\}$ is an undirected edge between vertices u and v

Undirected Graph Example

- Example: Let $G = (V, E)$ where:
  - $V = \{\text{San Franciso, Los Angeles, Denver, Chicago, Detroit, Washington, New York}\}$
  - E contains the following edges:
    - $\{\text{San Francisco, Los Angeles}\}$
    - $\{\text{San Francisco, Denver}\}$
    - $\{\text{Los Angeles, Denver}\}$
    - $\{\text{Denver, Chicago}\}$
    - $\{\text{Chicago, Detroit}\}$
    - $\{\text{Chicago, Washington}\}$
    - $\{\text{Chicago, New York}\}$
    - $\{\text{Detroit, New York}\}$

Undirected Graph Example

- Example continued:



Basic Terminology

- For the edge $e = \{u, v\}$, u and v are endpoints of e

- Two vertices u and v in an undirected graph G are adjacent (or neighbors) if u and v are endpoints of an edge e of G. Such an edge e is incident with the vertices u and v and connects u and v.

Basic Terminology

- It is possible for a graph to have two different edges between one pair of vertices. Such edges are called <u>parallel edges</u>. Two different edges are parallel if they connect the same two vertices.

- An undirected graph is <u>simple</u> if it has no parallel edges and it has no edges that connect a vertex to itself (self-loop)

Basic Terminology

- The set of all neighbors of a vertex v of $G = (V, E)$, denoted $N(v)$, is called the neighborhood of v.
- If A is a subset of V, then $N(A)$ is the set of vertices of G that are adjacent to at least one vertex in A

$$N(A) = \bigcup_{v \in A} N(v)$$

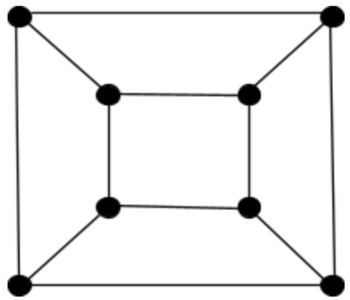
Basic Terminology

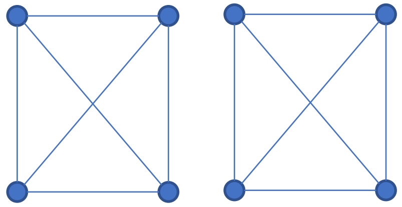
- The <u>degree of a vertex in an undirected graph</u> is the number of edges incident with it except that a loop at a vertex contributes twice to the degree of that vertex. The degree of vertex v is denoted by $\deg(v)$

- The <u>total degree</u> of an undirected graph is the sum of the degrees of its vertices

Basic Terminology

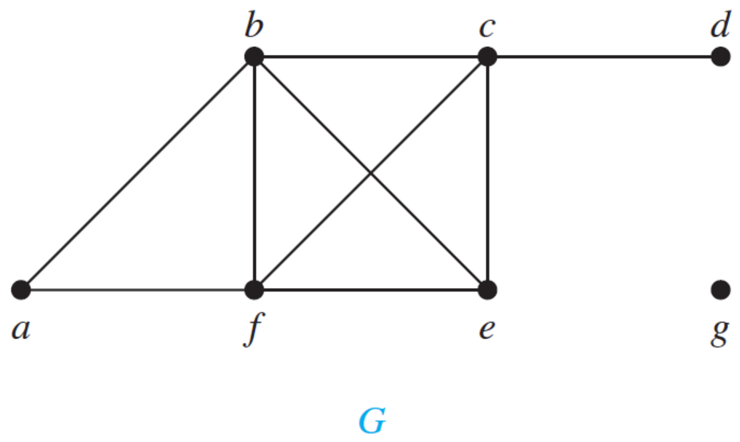
- An undirected graph is regular if each of its vertices has the same degree
- An undirected graph is d-regular if all of its vertices have degree d
- Example: two different 3-regular graphs each with 8 vertices





Basic Terminology

- Example 1: What are the degrees and neighborhoods of each vertex in the following graph?



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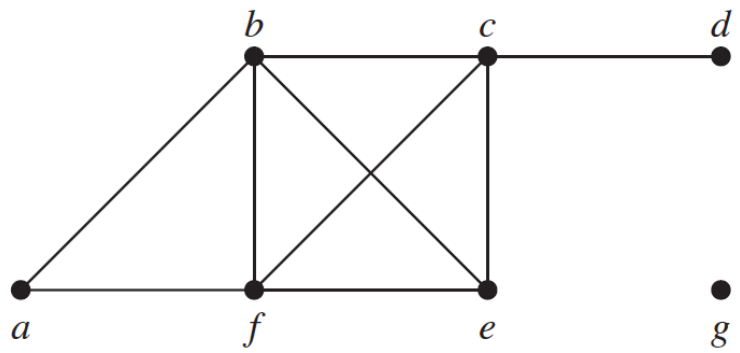


G

$\deg(a) = 2 \qquad N(a) = \{b, f\}$

$\deg(b) = 4 \qquad N(b) = \{a, c, e, f\}$

$\deg(c) = 4 \qquad N(c) = \{b, d, e, f\}$

$\deg(d) = 1 \qquad N(d) = \{c\}$

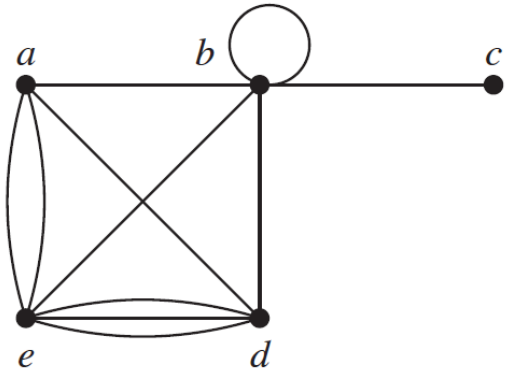
$\deg(e) = 3 \qquad N(e) = \{b, c, f\}$

$\deg(f) = 4 \qquad N(f) = \{a, b, c, e\}$

$\deg(g) = 0 \qquad N(g) = \emptyset$

Basic Terminology

- Example 1 continued: What are the degrees and neighborhoods of each vertex in the following graph?



H

Basic Terminology

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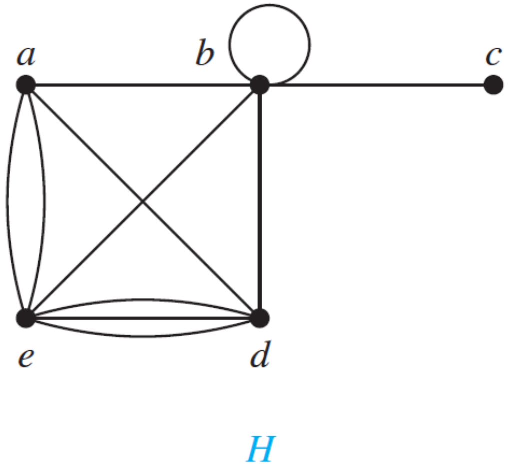


$\deg(a) = 4$ $\quad N(a) = \{b, d, e\}$

$\deg(b) = 6$ $\quad N(b) = \{a, b, c, d, e\}$

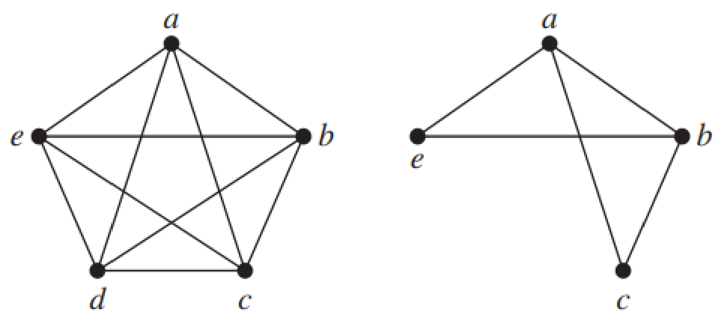
$\deg(c) = 1$ $\quad N(c) = \{b\}$

$\deg(d) = 5$ $\quad N(d) = \{a, b, e\}$

$\deg(e) = 6$ $\quad N(e) = \{a, b, d\}$

Basic Terminology

- A graph $G = (V_G, E_G)$ is a subgraph of $H = (V_H, E_H)$ if: a) $V_G \subseteq V_H$ and b) $E_G \subseteq E_H$



A graph and one of its sub graphs. What are some of its other subgraphs?

The Handshaking Theorem

- Let $G = (V, E)$ be an undirected graph with m edges. Then

$$\sum_{v \in V} \deg(v) = 2m$$

The Handshaking Theorem

- Proof by induction on the number of edges in the graph
    1. Base case. The graph has 0 edges

$$\sum_{v \in V} \deg(v) = 0 = 2 \cdot 0$$

The Handshaking Theorem

- Proof by induction on the number of edges in the graph

  2. Induction step.

     - The $\deg(v)$ function returns the degree of a vertex v. Usually, the graph to which vertex v belongs is implied by context
     - This proof refers to two graphs, G and H. For clarity $\deg_G(v)$ is the degree of v in graph G, and $\deg_H(v)$ is the degree of v in graph H

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- Proof by induction on the number of edges in the graph
  2. Induction step.
     1. If an undirected graph has k edges, then $\sum_{v \in V} \deg(v) = 2k$ Induction hypothesis
     2. A graph with $k+1$ edges has a subgraph with the same vertices and k edges. Let $G = (V_G, E_G)$ denote the subgraph and $H = (V_H, E_H)$ denote the original graph. $V_G = V_H$ and $E_G \subset E_H$. Let e be the $k+1$st edge: $E_G \cup \{e\} = E_H$

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     9. $\sum_{v \in V} \deg_H(v) = 2k + 2 = 2(k + 1)$

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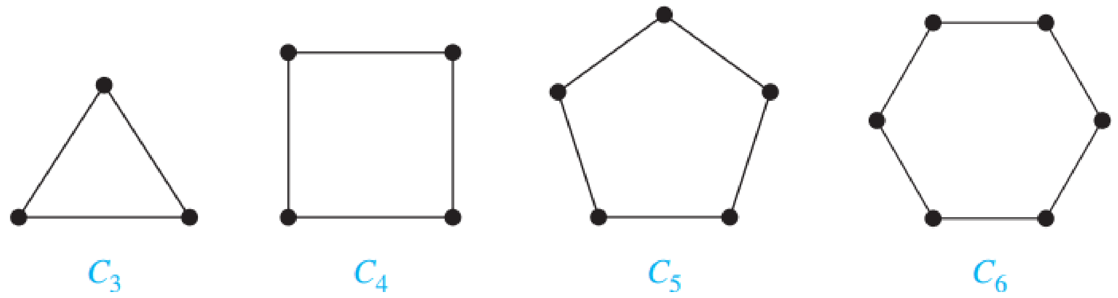
     15. $\sum_{v \in V} \deg_H(v) = 2k + 2 = 2(k+1)$
     16. In both cases, $\sum_{v \in V} \deg_H(v) = 2(k+1)$

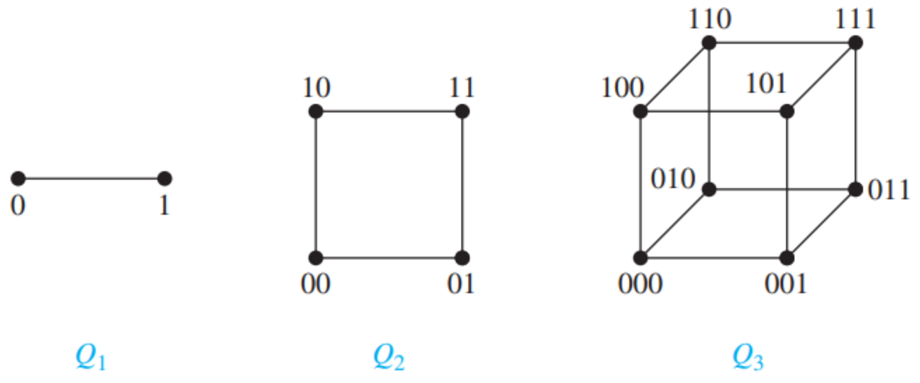
Common Graphs

- Some types of graphs occur frequently in the study of graphs
- A cycle (when referring to a graph) has edges that form exactly one cycle (as a walk) using all of the vertices of the graph. C_n denotes a cycle graph with n vertices. Note that it must be the case that $n \geq 3$



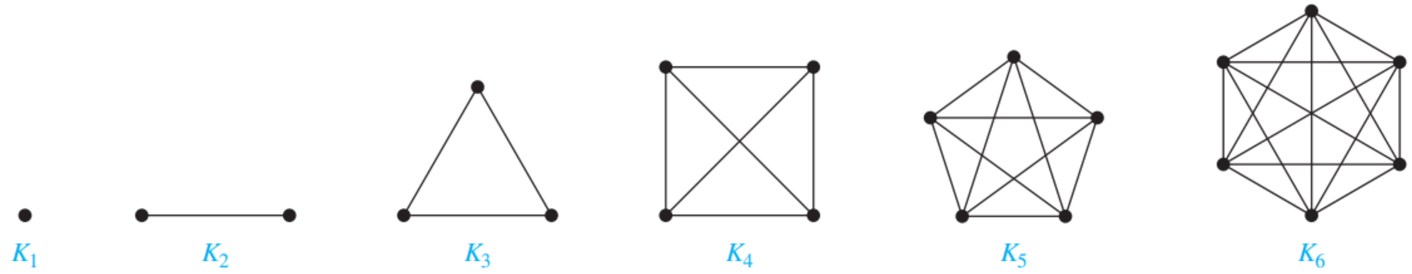
Common Graphs

- An n-dimensional hypercube, Q_n, has 2^n vertices representing the possible binary strings of length n. There is an edge between two vertices if their corresponding binary strings are different in only 1 place.



Common Graphs

- A <u>complete graph</u> has an edge between every pair of vertices. K_n denotes a complete graph with n vertices. K_n is sometimes called a <u>clique</u> of size n or an n-clique



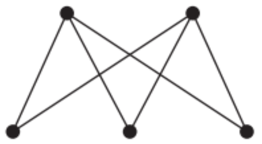
Common Graphs

- A <u>complete bipartite graph</u> $G = (V, E)$has a set if vertices that that can be divided into 2 nonempty sets V_1 and V_2 such that:
  - $V = V_1 \cup V_2$
  - $V_1 \cap V_2 = \emptyset$
  - $\{a, b\} \in E$ whenever a and b are in different vertex subsets
  - $\{a, b\} \notin E$ whenever a and b are in the same vertex subset

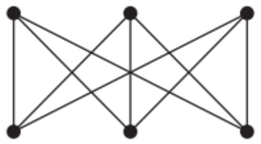
  $K_{m,n}$ denotes a complete bipartite graph where one vertex subset has m vertices and the other vertex subset has n vertices

Common Graphs

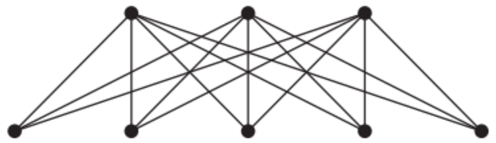
- Examples of complete bipartite graphs



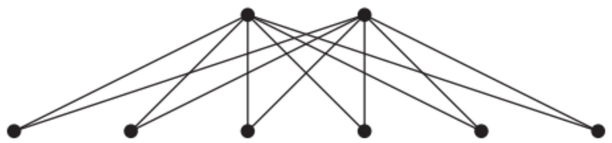
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



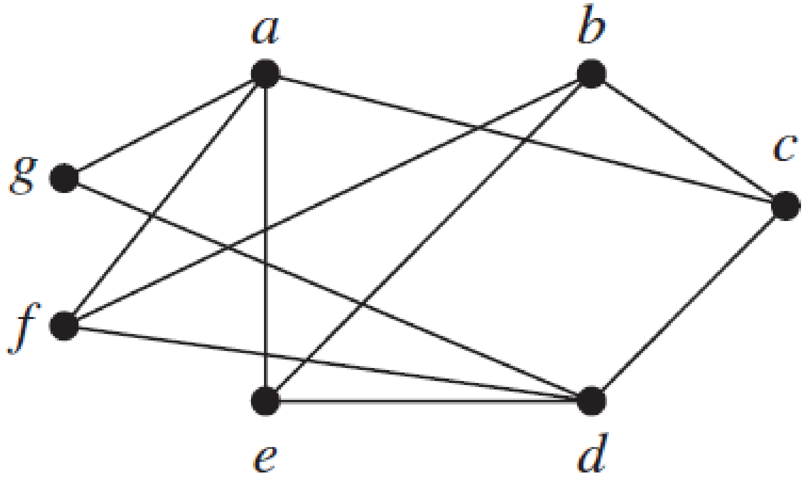
$K_{2,6}$

Common Graphs

- A complete bipartite graph is a special case of a bipartite graph. $G = (V, E)$ is bipartite if it has a set if vertices that that can be divided into 2 nonempty sets V_1 and V_2 such that:
  - $V = V_1 \cup V_2$
  - $V_1 \cap V_2 = \emptyset$
  - $\{a, b\} \notin E$ whenever a and b are in the same vertex subset

Bipartite Graphs

- Example: Is the graph below bipartite?



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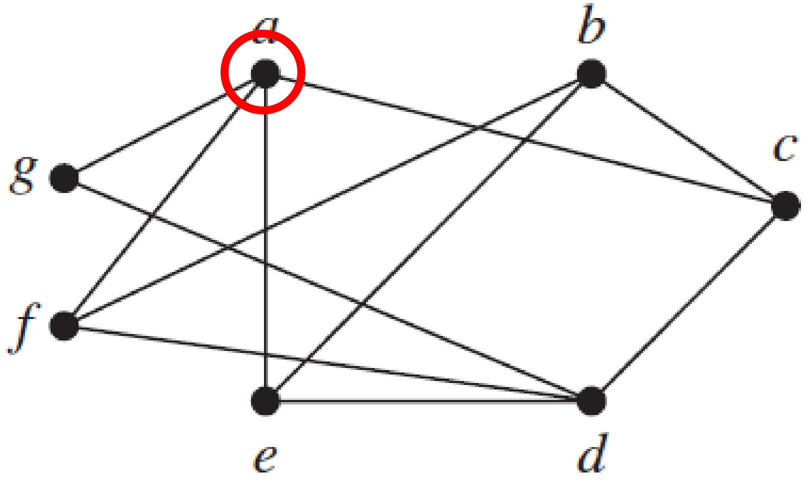
Bipartite Graphs

- Example: Is the graph below bipartite?
- Color vertex a red



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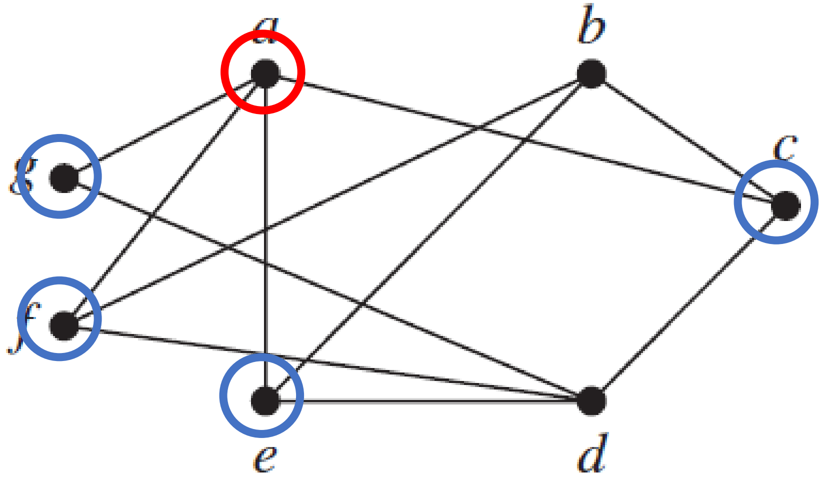
Bipartite Graphs

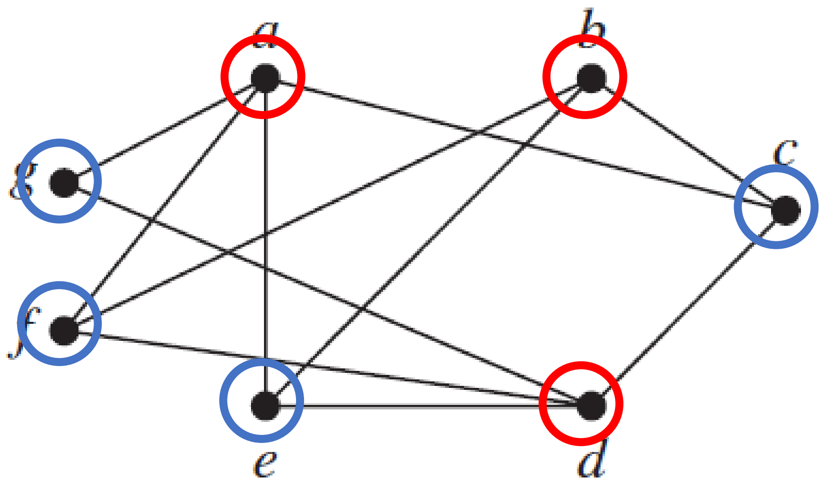
- Example: Is the graph below bipartite?
- Color the vertices adjacent to a blue



G

Bipartite Graphs

- Example: Is the graph below bipartite?
- Color the vertices adjacent to f red



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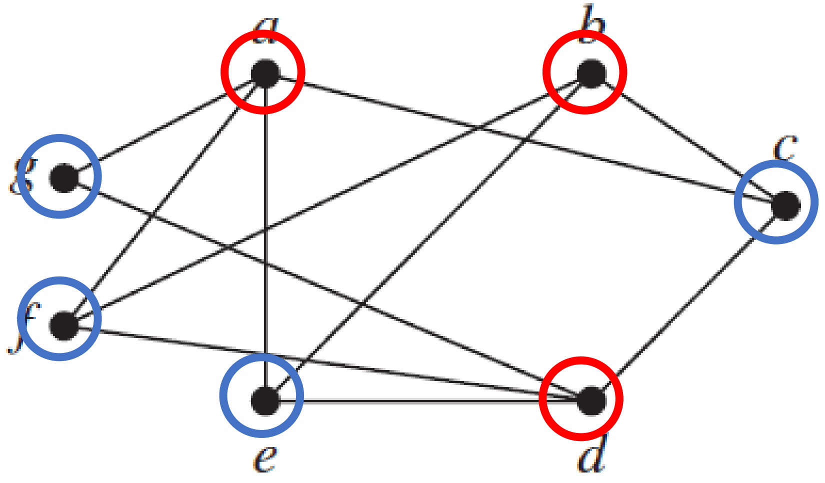
Bipartite Graphs

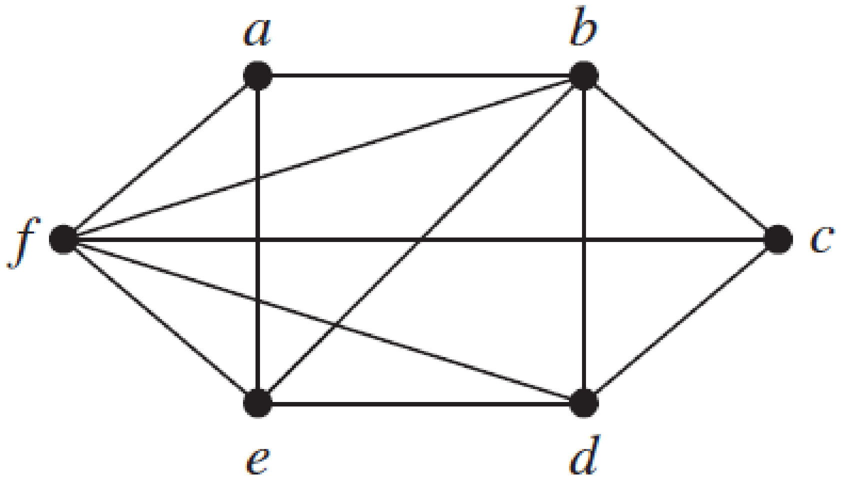
- Example: Is the graph below bipartite?
- There are no edges that connect two red vertices or two blue vertices, so the graph is bipartite



G

Bipartite Graphs

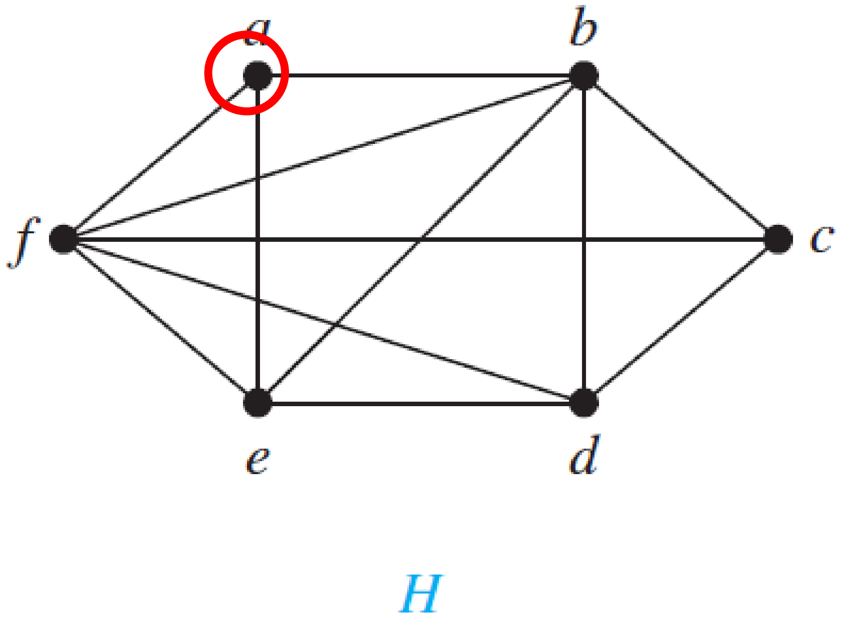
- Another example: Is the graph below bipartite?



H

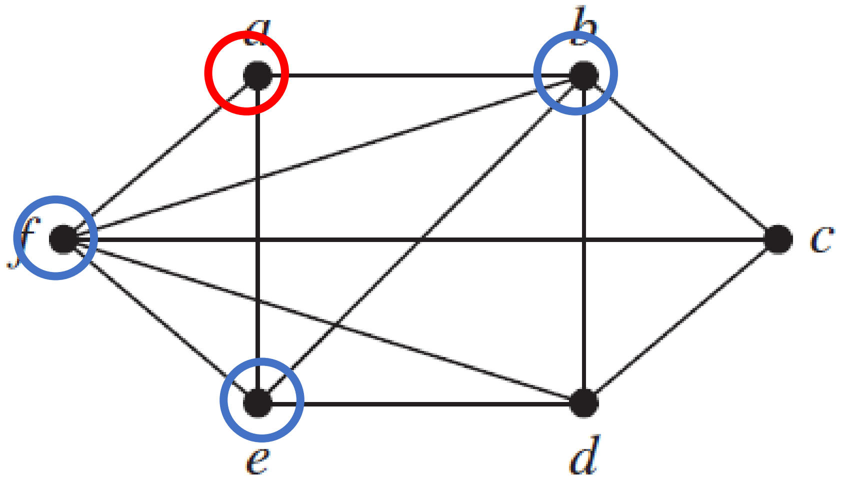
Bipartite Graphs

- Another example: Is the graph below bipartite?
- Color vertex a red



Bipartite Graphs

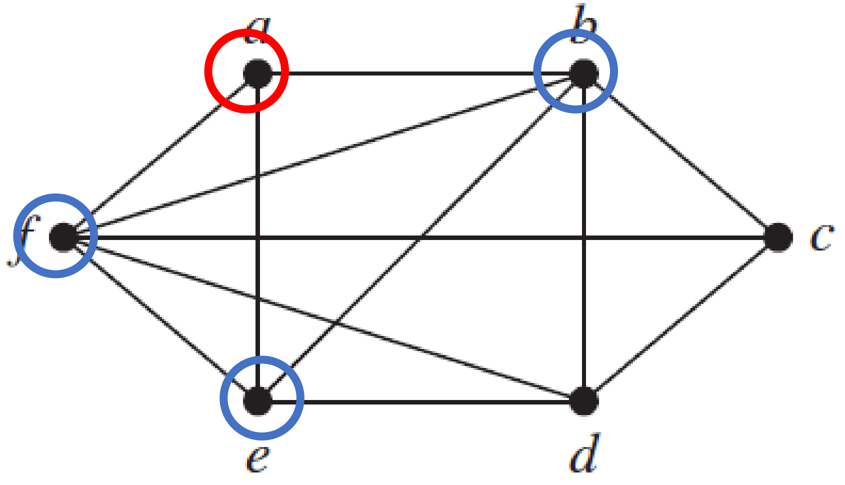
- Another example : Is the graph below bipartite?
- Color the vertices adjacent to a blue



H

Bipartite Graphs

- Another example : Is the graph below bipartite?
- Color the vertices adjacent to f red. This cannot be done, so the graph is not bipartite



H

Bipartite Graphs

- Theorem: A graph is bipartite if and only if it is possible to assign one of two different colors to each vertex so that no two adjacent vertices have the same color.

Bipartite Graphs

- Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex so that no two adjacent vertices have the same color.
- Proof by proving each implication:
  a) If a simple graph is bipartite then, then its vertices can be colored with two different colors so that no edge connects two vertices of the same color
  b) If the vertices of a simple graph can be colored with two different colors so that no edge connects vertices of the same color, then the graph is bipartite

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of a)

  1. Assume that $G = (V, E)$ is a bipartite simple graph.

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of a)

  1. Assume that $G = (V, E)$ is a bipartite simple graph.
  2. $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$, and no edge in E connects two vertices that are in V_1 or two vertices that are in V_2

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of a)

  1. Assume that $G = (V, E)$ is a bipartite simple graph.
  2. $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$, and no edge in E connects two vertices that are in V_1 or two vertices that are in V_2
  3. If each vertex in V_1 is colored red and each vertex in V_2 is colored blue, then no edge in E connects two vertices of the same color.

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of a)

  1. Assume that $G = (V, E)$ is a bipartite simple graph.
  2. $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$, and no edge in E connects two vertices that are in V_1 or two vertices that are in V_2
  3. If each vertex in V_1 is colored red and each vertex in V_2 is colored blue, then no edge in E connects two vertices of the same color.
  4. If $G = (V, E)$ is a bipartite simple graph, then its vertices can be colored with two different colors such that no edge connects two vertices of the same color

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of b)

  1. Assume $G = (V, E)$ is a graph with no edges connecting the same vertex and whose vertices are colored with two different colors such that no edge connects two vertices of the same color. Without loss of generalization assume that the colors are red and blue.

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of b)

  1. Assume $G = (V, E)$ is a graph with no edges connecting the same vertex and whose vertices are colored with two different colors such that no edge connects two vertices of the same color. Without loss of generalization assume that the colors are red and blue.
  2. $V = V_{red} \cup V_{blue}$ and $V_{red} \cap V_{blue} = \emptyset$ where the set of red vertices is V_{red} and the set of blue vertices is V_{blue}

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of b)

  1. Assume $G = (V, E)$ is a graph with no edges connecting the same vertex and whose vertices are colored with two different colors such that no edge connects two vertices of the same color. Without loss of generalization assume that the colors are red and blue.
  2. $V = V_{red} \cup V_{blue}$ and $V_{red} \cap V_{blue} = \emptyset$ where the set of red vertices is V_{red} and the set of blue vertices is V_{blue}
  3. $G = (V, E)$ is bipartite since there is no edge that connects two vertices in V_{red} or two vertices in V_{blue}

Bipartite Graphs

- Proof of Theorem 4 continued:

  Proof of b)

  1. Assume $G = (V, E)$ is a graph with no edges connecting the same vertex and whose vertices are colored with two different colors such that no edge connects two vertices of the same color. Without loss of generalization assume that the colors are red and blue.
  2. $V = V_{red} \cup V_{blue}$ and $V_{red} \cap V_{blue} = \emptyset$ where the set of red vertices is V_{red} and the set of blue vertices is V_{blue}
  3. $G = (V, E)$ is bipartite since there is no edge that connects two vertices in V_{red} or two vertices in V_{blue}
  4. If $G = (V, E)$ is a graph whose vertices are colored with two different colors such that no edge connects two vertices of the same color, then G is bipartite